EXERCISE 11.1 PAGE NO: 467

1. If a line makes angles 90°, 135°, 45° with the x, y and z-axes respectively, find its direction cosines.

Solution:

Let the direction cosines of the line be l, m and n.

Here let $\alpha = 90°$, $\beta = 135°$ and $\gamma = 45°$ So,

$l = \cos\alpha$, $m = \cos\beta$ and $n = \cos\gamma$

So direction cosines are l

$= \cos 90° = 0$

m $= \cos 135° = \cos(180° - 45°) = -\cos 45° = -1/\sqrt{2}$ n $= \cos 45° = 1/\sqrt{2}$

∴ The direction cosines of the line are $0, -1/\sqrt{2}, 1/\sqrt{2}$

2. Find the direction cosines of a line which makes equal angles with the coordinate axes.

Solution:

Given:

Angles are equal.

So let the angles be α, β, γ

Let the direction cosines of the line be l, m and n

$l = \cos\alpha$, $m = \cos\beta$ and $n = \cos\gamma$

Here given $\alpha = \beta = \gamma$ (Since, line makes equal angles with the coordinate axes) … (1)

The direction cosines are $l = \cos\alpha$, $m = \cos\beta$ and $n = \cos\gamma$ We have, $l^2 + m^2 + n^2 = 1$

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

From (1) we have,

$\cos^2\alpha + \cos^2\alpha + \cos^2\alpha = 1$

$3\cos^2\alpha = 1$

$\cos\alpha = \pm\sqrt{(1/3)}$

∴ The direction cosines are $l = \pm\sqrt{(1/3)}$, $m = \pm\sqrt{(1/3)}$, $n = \pm\sqrt{(1/3)}$

3. If a line has the direction ratios –18, 12, –4, then what are its direction cosines?

Solution:

Given

Direction ratios as -18, 12, -4

Where, a = -18, b = 12, c = -4

Let us consider the direction ratios of the line as a, b and c

Then the direction cosines are

$$\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}}$$

Where,

$$\begin{aligned}\sqrt{a^2+b^2+c^2} &= \sqrt{(-18)^2+12^2+(-4)^2}\\ &= \sqrt{324+144+16}\\ &= \sqrt{484}\\ &= 22\end{aligned}$$

∴ The direction cosines are

-18/22, 12/22, -4/22 => -9/11, 6/11, -2/11

4. Show that the points (2, 3, 4), (–1, –2, 1), (5, 8, 7) are collinear.

Solution:

If the direction ratios of two lines segments are proportional, then the lines are collinear.

Given:

A(2, 3, 4), B(−1, −2, 1), C(5, 8, 7)

Direction ratio of line joining A (2, 3, 4) and B (−1, −2, 1), are

(−1−2), (−2−3), (1−4) = (−3, −5, −3)

Where, a_1 = -3, b_1 = -5, c_1 = -3

Direction ratio of line joining B (−1, −2, 1) and C (5, 8, 7) are

(5− (−1)), (8− (−2)), (7−1) = (6, 10, 6)

Where, a_2 = 6, b_2 = 10 and c_2 =6

Hence it is clear that the direction ratios of AB and BC are of same proportions

By

$$\frac{a_1}{a_2} = \frac{-3}{6} = -2$$

$$\frac{b_1}{b_2} = \frac{-5}{10} = -2$$

And

$$\frac{c_1}{c_2} = \frac{-3}{6} = -2$$

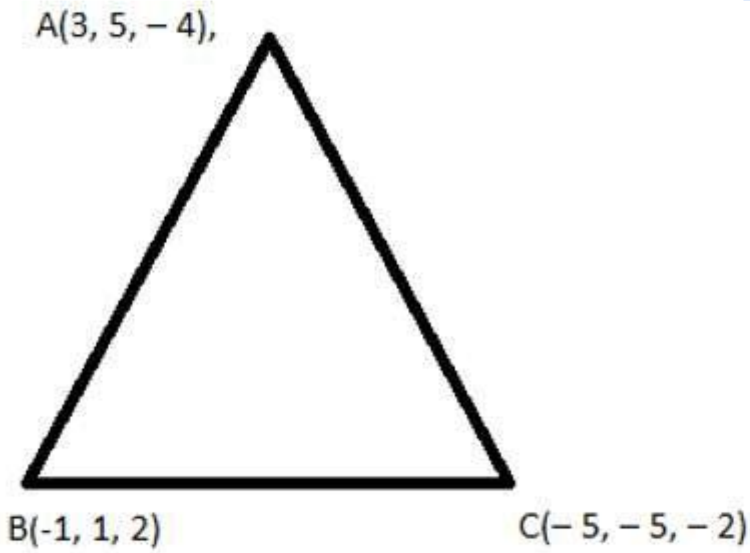
∴ A, B, C are collinear.

5. Find the direction cosines of the sides of the triangle whose vertices are (3, 5, –4), (-1, 1, 2) and (–5, –5, –2).

Solution:

Given:

The vertices are (3, 5, –4), (-1, 1, 2) and (–5, –5, –2).



The direction cosines of the two points passing through A(x_1, y_1, z_1) and B(x_2, y_2, z_2) is given by ($x_2 - x_1$), (y_2-y_1), (z_2-z_1)

Firstly let us find the direction ratios of AB

Where, A = (3, 5, -4) and B = (-1, 1, 2)

Ratio of AB = [$(x_2 - x_1)^2$, $(y_2 - y_1)^2$, $(z_2 - z_1)^2$]

= (-1-3), (1-5), (2-(-4)) = -4, -4, 6

Then by using the formula,

$\sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$

$\sqrt{[(-4)^2 + (-4)^2 + (6)^2]} = \sqrt{(16+16+36)}$

$= \sqrt{68}$

$= 2\sqrt{17}$

Now let us find the direction cosines of the line AB

By using the formula,

$$\frac{(x_2 - x_1)}{AB}, \frac{(y_2 - y_1)}{AB}, \frac{(z_2 - z_1)}{AB}$$

$-4/2\sqrt{17}$, $-4/2\sqrt{17}$, $6/2\sqrt{17}$

Or $-2/\sqrt{17}$, $-2/\sqrt{17}$, $3/\sqrt{17}$

Similarly,

Let us find the direction ratios of BC

Where, B = (-1, 1, 2) and C = (-5, -5, -2)

Ratio of AB = $[(x_2 - x_1)^2, (y_2 - y_1)^2, (z_2 - z_1)^2]$

$= (-5+1), (-5-1), (-2-2) = -4, -6, -4$

Then by using the formula,

$\sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$

$\sqrt{[(-4)^2 + (-6)^2 + (-4)^2]} = \sqrt{(16+36+16)}$

$= \sqrt{68}$

$= 2\sqrt{17}$

Now let us find the direction cosines of the line AB By using the formula,

$$\frac{(x_2 - x_1)}{AB}, \frac{(y_2 - y_1)}{AB}, \frac{(z_2 - z_1)}{AB}$$

$-4/2\sqrt{17}$, $-6/2\sqrt{17}$, $-4/2\sqrt{17}$

Or $-2/\sqrt{17}$, $-3/\sqrt{17}$, $-2/\sqrt{17}$

Similarly,

Let us find the direction ratios of CA

Where, C = (-5, -5, -2) and A = (3, 5, -4)

Ratio of AB = $[(x_2 - x_1)^2, (y_2 - y_1)^2, (z_2 - z_1)^2]$

$= (3+5), (5+5), (-4+2) = 8, 10, -2$

Then by using the formula,

$\sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$

$\sqrt{[(8)^2 + (10)^2 + (-2)^2]} = \sqrt{(64+100+4)}$

$= \sqrt{168}$

$= 2\sqrt{42}$

Now let us find the direction cosines of the line AB By using the formula,

$$\frac{(x_2 - x_1)}{AB}, \frac{(y_2 - y_1)}{AB}, \frac{(z_2 - z_1)}{AB}$$

$8/2\sqrt{42}, 10/2\sqrt{42}, -2/2\sqrt{42}$

Or $4/\sqrt{42}, 5/\sqrt{42}, -1/\sqrt{42}$

EXERCISE 11.2 PAGE NO: 477

1. Show that the three lines with direction cosines

$$\frac{12}{13}, \frac{-3}{13}, \frac{-4}{13}; \frac{4}{13}, \frac{12}{13}, \frac{3}{13}; \frac{3}{13}, \frac{-4}{13}, \frac{12}{13}$$ **Are mutually perpendicular.**

Solution:

Let us consider the direction cosines of L_1, L_2 and L_3 be l_1, m_1, n_1; l_2, m_2, n_2 and l_3, m_3, n_3.

We know that

If l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of two lines;

And θ is the acute angle between the two lines;

Then $\cos\theta = |l_1l_2 + m_1m_2 + n_1n_2|$

If two lines are perpendicular, then the angle between the two is $\theta = 90°$

For perpendicular lines, $| l_1l_2 + m_1m_2 + n_1n_2 | = \cos 90° = 0$, i.e. $| l_1l_2 + m_1m_2 + n_1n_2 | = 0$

So, in order to check if the three lines are mutually perpendicular, we compute $| l_1l_2 + m_1m_2 + n_1n_2 |$ for all the pairs of the three lines.

Firstly let us compute, $| l_1l_2 + m_1m_2 + n_1n_2 |$

$$|l_1l_2+m_1m_2+n_1n_2|=\left|\left(\frac{12}{13}\times\frac{4}{13}\right)+\left(\frac{-3}{13}\times\frac{12}{13}\right)+\left(\frac{-4}{13}\times\frac{3}{13}\right)\right|=\frac{48}{13}+\left(\frac{-36}{13}\right)+\left(\frac{-12}{13}\right)$$

$$=\frac{48+(-48)}{13}=0$$

So, $L_1 \perp L_2$ …… (1)

Similarly,

Let us compute, $| l_2l_3 + m_2m_3 + n_2n_3 |$

$$|l_2l_3+m_2m_3+n_2n_3|=\left|\left(\frac{4}{13}\times\frac{3}{13}\right)+\left(\frac{12}{13}\times\frac{-4}{13}\right)+\left(\frac{3}{13}\times\frac{12}{13}\right)\right|=\frac{12}{13}+\left(\frac{-48}{13}\right)+\frac{36}{13}$$

$$=\frac{(-48)+48}{13}=0$$

So, $L_2 \perp L_3$ ….. (2)

Similarly,

Let us compute, $| l_3l_1 + m_3m_1 + n_3n_1 |$

$$|l_3l_1+m_3m_1+n_3n_1|=\left|\left(\frac{3}{13}\times\frac{12}{13}\right)+\left(\frac{-4}{13}\times\frac{-3}{13}\right)+\left(\frac{12}{13}\times\frac{-4}{13}\right)\right|=\frac{36}{13}+\frac{12}{13}+\left(\frac{-48}{13}\right)$$

$$=\frac{48+(-48)}{13}=0$$

So, $L_1 \perp L_3$ ….. (3)

∴ By (1), (2) and (3), the lines are perpendicular. L_1, L_2 and L_3 are mutually perpendicular.

2. Show that the line through the points (1, –1, 2), (3, 4, –2) is perpendicular to the line through the points (0, 3, 2) and (3, 5, 6).

Solution:

Given:

The points (1, –1, 2), (3, 4, –2) and (0, 3, 2), (3, 5, 6).

Let us consider AB be the line joining the points, (1, -1, 2) and (3, 4, -2), and CD be the line through the points (0, 3, 2) and (3, 5, 6).
Now,
The direction ratios, a_1, b_1, c_1 of AB are (3 - 1), (4 – (-1)), (-2 - 2) = 2, 5, -4.
Similarly,
The direction ratios, a_2, b_2, c_2 of CD are (3 - 0), (5 - 3), (6 - 2) = 3, 2, 4.

Then, AB and CD will be perpendicular to each other, if $a_1a_2 + b_1b_2 + c_1c_2 = 0$ $a_1a_2 + b_1b_2 + c_1c_2 = 2(3) + 5(2) + 4(-4)$
$= 6 + 10 – 16$
$= 0$
∴ AB and CD are perpendicular to each other.

3. Show that the line through the points (4, 7, 8), (2, 3, 4) is parallel to the line through the points (–1, –2, 1), (1, 2, 5).
Solution:
Given:
The points (4, 7, 8), (2, 3, 4) and (–1, –2, 1), (1, 2, 5).
Let us consider AB be the line joining the points, (4, 7, 8), (2, 3, 4) and CD be the line through the points (–1, –2, 1), (1, 2, 5).
Now,
The direction ratios, a_1, b_1, c_1 of AB are (2 - 4), (3 - 7), (4 - 8) = -2, -4, -4.

The direction ratios, a_2, b_2, c_2 of CD are (1 – (-1)), (2 – (-2)), (5 - 1) = 2, 4, 4.

Then AB will be parallel to CD, if
$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$
So, $a_1/a_2 = -2/2 = -1$
$b_1/b_2 = -4/4 = -1$ $c_1/c_2 = -4/4 = -1$

∴ We can say that,

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

-1 = -1 = -1

Hence, AB is parallel to CD where the line through the points (4, 7, 8), (2, 3, 4) is parallel to the line through the points (–1, –2, 1), (1, 2, 5)

4. Find the equation of the line which passes through the point (1, 2, 3) and is parallel to the vector $3\hat{i} + 2\hat{j} - 2\hat{k}$.

Solution:

Given:

Line passes through the point (1, 2, 3) and is parallel to the vector.

We know that

Vector equation of a line that passes through a given point whose position vector is $\vec{a}$ and parallel to a given vector $\vec{b}$ is

$\vec{r} = \vec{a} + \lambda \vec{b}$.

So, here the position vector of the point (1, 2, 3) is given by

$\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$ and the parallel vector is $3\hat{i} + 2\hat{j} - 2\hat{k}$

∴ The vector equation of the required line is:

$\vec{r} = \hat{i} + 2\hat{j} + 3\hat{k} + \lambda\left(3\hat{i} + 2\hat{j} - 2\hat{k}\right)$,

Where λ is constant.

5. Find the equation of the line in vector and in Cartesian form that passes through the point with position vector $2\hat{i} - \hat{j} + 4\hat{k}$ and $\hat{i} + 2\hat{j} - \hat{k}$. is in the direction

Solution:

It is given that
Vector equation of a line that passes through a given point whose position vector is $\vec{a}$ and parallel to a given vector $\vec{b}$ is $\vec{r} = \vec{a} + \lambda \vec{b}$
Here let, $\vec{a} = 2\hat{i} - \hat{j} + 4\hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$
So, the vector equation of the required line is:

$$\vec{r} = 2\hat{i} - \hat{j} + 4\hat{k} + \lambda\left(\hat{i} + 2\hat{j} - \hat{k}\right)$$

Now the Cartesian equation of a line through a point (x_1, y_1, z_1) and having direction cosines l, m, n is given by

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

We know that if the direction ratios of the line are a, b, c, then

$$l = \frac{a}{\sqrt{a^2 + b^2 + c^2}},\ m = \frac{b}{\sqrt{a^2 + b^2 + c^2}},\ n = \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

The Cartesian equation of a line through a point (x_1, y_1, z_1) and having direction ratios a, b, c is

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

Here, $x_1 = 2$, $y_1 = -1$, $z_1 = 4$ and $a = 1$, $b = 2$, $c = -1$

∴ The Cartesian equation of the required line is:

$$\frac{x - 2}{1} = \frac{y - (-1)}{2} = \frac{z - 4}{-1} \Rightarrow \frac{x - 2}{1} = \frac{y + 1}{2} = \frac{z - 4}{-1}$$

6. Find the Cartesian equation of the line which passes through the point (–2, 4, –5) and parallel to the line given by

$$\frac{x + 3}{3} = \frac{y - 4}{5} = \frac{z + 8}{6}.$$

Solution:
Given:
The points (-2, 4, -5)

We know that

The Cartesian equation of a line through a point (x_1, y_1, z_1) and having direction ratios a, b, c is

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$$

Here, the point (x_1, y_1, z_1) is (-2, 4, -5) and the direction ratio is given by:

a = 3, b = 5, c = 6

∴ The Cartesian equation of the required line is:

$$\frac{x-(-2)}{3}=\frac{y-4}{5}=\frac{z-(-5)}{6} \Rightarrow \frac{x+2}{3}=\frac{y-4}{5}=\frac{z+5}{6}$$

7. The Cartesian equation of a line is

$$\frac{x-5}{3}=\frac{y+4}{7}=\frac{z-6}{2}$$ **. Write its vector form.**

Solution:

Given:

The Cartesian equation is

$$\frac{x-5}{3}=\frac{y+4}{7}=\frac{z-6}{2} \quad \ldots.(1)$$

We know that

The Cartesian equation of a line passing through a point (x_1, y_1, z_1) and having direction cosines l, m, n is

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$$

So when comparing this standard form with the given equation, we get

$x_1 = 5$, $y_1 = -4$, $z_1 = 6$ and $l = 3$, $m = 7$, $n = 2$

The point through which the line passes has the position vector
$\vec{a} = 5\hat{i} - 4\hat{j} + 6\hat{k}$ and
The vector parallel to the line is given by $\vec{b} = 3\hat{i} + 7\hat{j} + 2\hat{k}$

Since, vector equation of a line that passes through a given point whose position vector is $\vec{a}$ and parallel to a given vector $\vec{b}$ is $\vec{r} = \vec{a} + \lambda \vec{b}$

∴ The required line in vector form is given as:

$$\vec{r} = \left(5\hat{i} - 4\hat{j} + 6\hat{k}\right) + \lambda\left(3\hat{i} + 7\hat{j} + 2\hat{k}\right)$$

8. Find the vector and the Cartesian equations of the lines that passes through the origin and (5, –2, 3). Solution:

Given:

The origin (0, 0, 0) and the point (5, -2, 3)

We know that

The vector equation of as line which passes through two points whose position vectors are $\vec{a}$ and $\vec{b}$ is $\vec{r} = \vec{a} + \lambda\left(\vec{b} - \vec{a}\right)$

Here, the position vectors of the two points (0, 0, 0) and (5, -2, 3) are $\vec{a} = 0\hat{i} + 0\hat{j} + 0\hat{k}$ and $\vec{b} = 5\hat{i} - 2\hat{j} + 3\hat{k}$, respectively.

∴ The vector equation of the required line is given as:

$$\vec{r} = 0\hat{i} + 0\hat{j} + 0\hat{k} + \lambda\left[\left(5\hat{i} - 2\hat{j} + 3\hat{k}\right) - \left(0\hat{i} + 0\hat{j} + 0\hat{k}\right)\right]$$

$$\vec{r} = \lambda\left(5\hat{i} - 2\hat{j} + 3\hat{k}\right)$$

Now, by using the formula,

Cartesian equation of a line that passes through two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given as

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

So, the Cartesian equation of the line that passes through the origin (0, 0, 0) and (5, – 2, 3) is

$$\frac{x - 0}{5 - 0} = \frac{y - 0}{-2 - 0} = \frac{z - 0}{3 - 0} \Rightarrow \frac{x}{5} = \frac{y}{-2} = \frac{z}{3}$$

∴ The vector equation is

$$\vec{r} = \lambda\left(5\hat{i} - 2\hat{j} + 3\hat{k}\right)$$

The Cartesian equation is

$$\frac{x}{5} = \frac{y}{-2} = \frac{z}{3}$$

9. Find the vector and the Cartesian equations of the line that passes through the points (3, –2, –5), (3, –2, 6).

Solution:

Given:
The points (3, -2, -5) and (3, -2, 6)
Firstly let us calculate the vector form:
The vector equation of as line which passes through two points whose position vectors are $\vec{a}$ and $\vec{b}$ is $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$

Here, the position vectors of the two points (3, –2, –5) and (3, –2, 6) are $\vec{a} = 3\hat{i} - 2\hat{j} - 5\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + 6\hat{k}$ respectively.

∴ The vector equation of the required line is:

$$\vec{r} = 3\hat{i} - 2\hat{j} - 5\hat{k} + \lambda\left[\left(3\hat{i} - 2\hat{j} + 6\hat{k}\right) - \left(3\hat{i} - 2\hat{j} - 5\hat{k}\right)\right]$$

$$\vec{r} = 3\hat{i} - 2\hat{j} - 5\hat{k} + \lambda\left(3\hat{i} - 2\hat{j} + 6\hat{k} - 3\hat{i} + 2\hat{j} + 5\hat{k}\right)$$

$$\vec{r} = 3\hat{i} - 2\hat{j} - 5\hat{k} + \lambda\left(11\hat{k}\right)$$

Now,
By using the formula,
Cartesian equation of a line that passes through two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

So, the Cartesian equation of the line that passes through the origin (3, -2, -5) and (3, -2, 6) is

$$\frac{x - 3}{3 - 3} = \frac{y - (-2)}{(-2) - (-2)} = \frac{z - (-5)}{6 - (-5)}$$

$$\frac{x - 3}{0} = \frac{y + 2}{0} = \frac{z + 5}{11}$$

∴ The vector equation is

$$\vec{r} = 3\hat{i} - 2\hat{j} - 5\hat{k} + \lambda\left(11\hat{k}\right)$$

The Cartesian equation is

$$\frac{x - 3}{0} = \frac{y + 2}{0} = \frac{z + 5}{11}$$

10. Find the angle between the following pairs of lines:

(i) $\vec{r} = 2\hat{i} - 5\hat{j} + \hat{k} + \lambda(3\hat{i} + 2\hat{j} + 6\hat{k})$ and

$\vec{r} = 7\hat{i} - 6\hat{k} + \mu(\hat{i} + 2\hat{j} + 2\hat{k})$

(ii) $\vec{r} = 3\hat{i} + \hat{j} - 2\hat{k} + \lambda(\hat{i} - \hat{j} - 2\hat{k})$ and

$\vec{r} = 2\hat{i} - \vec{j} - 56\hat{k} + \mu(3\hat{i} - 5\hat{j} - 4\hat{k})$

Solution:

Let us consider θ be the angle between the given lines.

If θ is the acute angle between $\vec{r} = \overrightarrow{a_1} + \lambda \overrightarrow{b_1}$ and $\vec{r} = \overrightarrow{a_2} + \mu \overrightarrow{b_2}$ then

$$\cos\theta = \left| \frac{\vec{b_1}\vec{b_2}}{\left|\vec{b_1}\right| \left|\vec{b_2}\right|} \right| \quad \ldots\ldots (1)$$

(i) $\vec{r} = 2\hat{i} - 5\hat{j} + \hat{k} + \lambda(3\hat{i} + 2\hat{j} + 6\hat{k})$ and

$\vec{r} = 7\hat{i} - 6\hat{k} + \mu(\hat{i} + 2\hat{j} + 2\hat{k})$

Here $\overrightarrow{b_1} = 3\hat{i} + 2\hat{j} + 6\hat{k}$ and $\overrightarrow{b_2} = \hat{i} + 2\hat{j} + 2\hat{k}$

So, from equation (1), we have

$$\cos\theta = \left| \frac{\left(3\hat{i} + 2\hat{j} + 6\hat{k}\right).\left(\hat{i} + 2\hat{j} + 2\hat{k}\right)}{\left|3\hat{i} + 2\hat{j} + 6\hat{k}\right|.\left|\hat{i} + 2\hat{j} + 2\hat{k}\right|} \right| \quad \ldots\ldots (2)$$

We know that,

$$\left|a\hat{i} + b\hat{j} + c\hat{k}\right| = \sqrt{a^2 + b^2 + c^2}$$

So,

$$\left|3\hat{i} + 2\hat{j} + 6\hat{k}\right| = \sqrt{3^2 + 2^2 + 6^2} = \sqrt{9 + 4 + 36} = \sqrt{49} = 7$$

And

$$\left|\hat{i} + 2\hat{j} + 2\hat{k}\right| = \sqrt{1^2 + 2^2 + 2^2} = \sqrt{1 + 4 + 4} = \sqrt{9} = 3$$

Now, we know that

$$\left(a_1\hat{i} + b_1\hat{j} + c_1\hat{k}\right).\left(a_2\hat{i} + b_2\hat{j} + c_2\hat{k}\right) = a_1a_2 + b_1b_2 + c_1c_2$$

So,

$$\left(3\hat{i}+2\hat{j}+6\hat{k}\right).\left(\hat{i}+2\hat{j}+2\hat{k}\right)=3\times1+2\times2+6\times2=3+4+12=19$$

By (2), we have

$$\cos\theta = \frac{19}{7\times3} = \frac{19}{21}$$

$$\theta = \cos^{-1}\left(\frac{19}{21}\right)$$

(ii) $\vec{r} = 3\hat{i}+\hat{j}-2\hat{k}+\lambda(\hat{i}-\hat{j}-2\hat{k})$ and

$\vec{r} = 2\hat{i}-\hat{j}-56\hat{k}+\mu(3\hat{i}-5\hat{j}-4\hat{k})$

Here, $\overrightarrow{b_1} = \hat{i}-\hat{j}-2\hat{k}$ and $\overrightarrow{b_2} = 3\hat{i}-5\hat{j}-4\hat{k}$

So, from (1), we have

$$\cos\theta = \left|\frac{\left(\hat{i}-\hat{j}-2\hat{k}\right).\left(3\hat{i}-5\hat{j}-4\hat{k}\right)}{\left|\hat{i}-\hat{j}-2\hat{k}\right|\left|3\hat{i}-5\hat{j}-4\hat{k}\right|}\right| \quad \ldots.(3)$$

We know that,

$$\left|a\hat{i}+b\hat{j}+c\hat{k}\right| = \sqrt{a^2+b^2+c^2}$$

So,

$$\left|\hat{i}-\hat{j}-2\hat{k}\right| = \sqrt{1^2+(-1)^2+2^2} = \sqrt{1+1+4} = \sqrt{6} = \sqrt{3}\times\sqrt{2}$$

And

$$\left|3\hat{i}-5\hat{j}-4\hat{k}\right| = \sqrt{3^2+(-5)^2+(-4)^2} = \sqrt{9+25+16} = \sqrt{50} = 5\sqrt{2}$$

Now, we know that

$$\left(a_1\hat{i}+b_1\hat{j}+c_1\hat{k}\right).\left(a_2\hat{i}+b_2\hat{j}+c_2\hat{k}\right)=a_1a_2+b_1b_2+c_1c_2$$

$$\therefore\ \left(\hat{i}-\hat{j}-2\hat{k}\right).\left(3\hat{i}-5\hat{j}-4\hat{k}\right)=1\times3+(-1)\times(-5)+(-2)\times(-4)=3+5+8=16$$

By (3), we have

$$\cos\theta = \frac{16}{\sqrt{3}\times\sqrt{2}\times 5\sqrt{2}} = \frac{16}{5\times 2\sqrt{3}} = \frac{8}{5\sqrt{3}}$$

$$\theta = \cos^{-1}\left(\frac{8}{5\sqrt{3}}\right)$$

11. Find the angle between the following pair of lines:

(i) $\frac{x-2}{2} = \frac{y-1}{5} - \frac{z+3}{-3}$ and $\frac{x+2}{-1} = \frac{y-4}{8} - \frac{z-5}{4}$

(ii) $\frac{x}{2} = \frac{y}{2} = \frac{z}{1}$ and $\frac{x-5}{4} = \frac{y-2}{1} = \frac{z-3}{8}$

Solution:

We know that

If

$$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1} \text{ and } \frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$$ are the equations of two lines, then the acute angle between the two lines is given by

$\cos\theta = | l_1l_2 + m_1m_2 + n_1n_2 |$ …… (1)

(i) $\frac{x-2}{2}=\frac{y-1}{5}-\frac{z+3}{-3}$ and $\frac{x+2}{-1}=\frac{y-4}{8}-\frac{z-5}{4}$

Here, $a_1 = 2$, $b_1 = 5$, $c_1 = -3$ and

$a_2 = -1$, $b_2 = 8$, $c_2 = 4$

Now,

$$l=\frac{a}{\sqrt{a^2+b^2+c^2}},\ m=\frac{b}{\sqrt{a^2+b^2+c^2}},\ n=\frac{c}{\sqrt{a^2+b^2+c^2}} \quad \text{...... (2)}$$

Here, we know that

$$\sqrt{a_1^2+b_1^2+c_1^2}=\sqrt{2^2+5^2+(-3)^2}=\sqrt{4+25+9}=\sqrt{38}$$

And

$$\sqrt{a_2^2+b_2^2+c_2^2}=\sqrt{(-1)^2+8^2+4^2}=\sqrt{1+64+16}=\sqrt{81}=9$$

So, from equation (2), we have

$$l_1 = \frac{a_1}{\sqrt{a_1^2+b_1^2+c_1^2}} = \frac{2}{\sqrt{38}},\ m_1 = \frac{b_1}{\sqrt{a_1^2+b_1^2+c_1^2}} = \frac{5}{\sqrt{38}},\ n_1 = \frac{c_1}{\sqrt{a_1^2+b_1^2+c_1^2}} = \frac{-3}{\sqrt{38}}$$

And

$$l_2 = \frac{a_2}{\sqrt{a_2^2+b_2^2+c_2^2}} = \frac{-1}{9},\ m_2 = \frac{b_2}{\sqrt{a_2^2+b_2^2+c_2^2}} = \frac{8}{9},\ n_2 = \frac{c_2}{\sqrt{a_2^2+b_2^2+c_2^2}} = \frac{4}{9}$$

∴ From equation (1), we have

$$\cos\theta = \left|\left(\frac{2}{\sqrt{38}}\right)\times\left(\frac{-1}{9}\right)+\left(\frac{5}{\sqrt{38}}\right)\times\left(\frac{8}{9}\right)+\left(\frac{-3}{\sqrt{38}}\right)\times\left(\frac{4}{9}\right)\right|$$

$$= \left|\frac{-2+40-12}{9\sqrt{38}}\right| = \left|\frac{40-12}{9\sqrt{38}}\right| = \frac{26}{9\sqrt{38}}$$

$$\theta = \cos^{-1}\left(\frac{26}{9\sqrt{38}}\right)$$

(ii) $\frac{x}{2} = \frac{y}{2} = \frac{z}{1}$ and $\frac{x-5}{4} = \frac{y-2}{1} = \frac{z-3}{8}$

Here, $a_1 = 2$, $b_1 = 2$, $c_1 = 1$ and

$a_2 = 4$, $b_2 = 1$, $c_2 = 8$

Here, we know that

$$\sqrt{a_1^2+b_1^2+c_1^2} = \sqrt{2^2+2^2+1^2} = \sqrt{4+4+1} = \sqrt{9} = 3$$

And

$$\sqrt{a_2^2+b_2^2+c_2^2} = \sqrt{4^2+1^2+8^2} = \sqrt{16+1+64} = \sqrt{81} = 9$$

So, from equation (2), we have

$$l_1 = \frac{a_1}{\sqrt{a_1^2+b_1^2+c_1^2}} = \frac{2}{3},\ m_1 = \frac{b_1}{\sqrt{a_1^2+b_1^2+c_1^2}} = \frac{2}{3},\ n_1 = \frac{c_1}{\sqrt{a_1^2+b_1^2+c_1^2}} = \frac{1}{3}$$

And

$$l_2 = \frac{a_2}{\sqrt{a_2^2+b_2^2+c_2^2}} = \frac{4}{9},\ m_2 = \frac{b_2}{\sqrt{a_2^2+b_2^2+c_2^2}} = \frac{1}{9},\ n_2 = \frac{c_2}{\sqrt{a_2^2+b_2^2+c_2^2}} = \frac{8}{9}$$

∴ From equation (1), we have

$$\cos\theta = \left|\left(\frac{2}{3}\times\frac{4}{9}\right)+\left(\frac{2}{3}\times\frac{1}{9}\right)+\left(\frac{1}{3}\times\frac{8}{9}\right)\right| = \left|\frac{8+2+8}{27}\right| = \frac{18}{27} = \frac{2}{3}$$

$$\theta = \cos^{-1}\left(\frac{2}{3}\right)$$

12. Find the values of p so that the lines
$$\frac{1-x}{3} = \frac{7y-14}{2p} = \frac{z-3}{2} \text{ and } \frac{7-7x}{3p} = \frac{y-5}{1} = \frac{6-z}{5}$$
are at right angles.

Solution:

The standard form of a pair of Cartesian lines is:

$$\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1} \text{ and } \frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2} \quad \text{.... (1)}$$

So the given equations can be written according to the standard form, i.e.

$$\frac{-(x-1)}{3} = \frac{7(y-2)}{2p} = \frac{z-3}{2} \text{ and } \frac{-7(x-1)}{3p} = \frac{y-5}{1} = \frac{-(z-6)}{5}$$

$$\frac{x-1}{-3} = \frac{y-2}{2p/7} = \frac{z-3}{2} \text{ and } \frac{x-1}{-3p/7} = \frac{y-5}{1} = \frac{z-6}{-5} \quad \text{.... (2)}$$

Now, comparing equation (1) and (2), we get

$$a_1 = -3,\ b_1 = \frac{2p}{7},\ c_1 = 2 \text{ and } a_2 = \frac{-3p}{7},\ b_2 = 1,\ c_2 = -5$$

So the direction ratios of the lines are

-3, 2p/7, 2 and -3p/7, 1, -5

Now, as both the lines are at right angles,

So, $a_1a_2 + b_1b_2 + c_1c_2 = 0$

(-3) (-3p/7) + (2p/7) (1) + 2 (-5) = 0

9p/7 + 2p/7 – 10 = 0

(9p+2p)/7 = 10

11p/7 = 10
11p = 70 p
= 70/11
∴ The value of p is 70/11

13. Show that the lines

$$\frac{x-5}{7}=\frac{y+2}{-5}=\frac{z}{1} \text{ and } \frac{x}{1}=\frac{y}{2}=\frac{z}{3}$$ **are perpendicular to each other.**

Solution:

The equations of the given lines are

$$\frac{x-5}{7}=\frac{y+2}{-5}=\frac{z}{1} \text{ and } \frac{x}{1}=\frac{y}{2}=\frac{z}{3}$$

Two lines with direction ratios is given as

$a_1a_2 + b_1b_2 + c_1c_2 = 0$

So the direction ratios of the given lines are 7, -5, 1 and 1, 2, 3

i.e., $a_1 = 7$, $b_1 = -5$, $c_1 = 1$ and $a_2 = 1$, $b_2 = 2$, $c_2 = 3$ Now,

Considering

$a_1a_2 + b_1b_2 + c_1c_2 = 7 \times 1 + (-5) \times 2 + 1 \times 3$
$= 7 - 10 + 3$
$= -3 + 3$
$= 0$

∴ The two lines are perpendicular to each other.

14. Find the shortest distance between the lines

$$\vec{r} = (\hat{i} + 2\hat{j} + \hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k}) \text{ and}$$

$$\vec{r} = 2\hat{i} - \hat{j} - \hat{k} + \mu(2\hat{i} + \hat{j} + 2\hat{k})$$

Solution:

We know that the shortest distance between two lines $\vec{r} = \overrightarrow{a_1} + \lambda \overrightarrow{b_1}$ and $\vec{r} = \overrightarrow{a_2} + \mu \overrightarrow{b_2}$ is given as:

$$d = \left| \frac{\left(\overrightarrow{b_1} \times \overrightarrow{b_2}\right).\left(\overrightarrow{a_2} - \overrightarrow{a_1}\right)}{\left|\overrightarrow{b_1} \times \overrightarrow{b_2}\right|} \right| \quad \text{.... (1)}$$

Here by comparing the equations we get,

$\overrightarrow{a_1} = \hat{i} + 2\hat{j} + \hat{k}$, $\overrightarrow{b_1} = \hat{i} - \hat{j} + \hat{k}$ and

$\overrightarrow{a_2} = 2\hat{i} - \hat{j} - \hat{k}$, $\overrightarrow{b_2} = 2\hat{i} + \hat{j} + 2\hat{k}$

Now,

$$\left(x_1\hat{i}+y_1\hat{j}+z_1\hat{k}\right)-\left(x_2\hat{i}+y_2\hat{j}+z_2\hat{k}\right)=(x_1-x_2)\hat{i}+(y_1-y_2)\hat{j}+(z_1-z_2)\hat{k}$$

$$\overrightarrow{a_2}-\overrightarrow{a_1}=\left(2\hat{i}-\hat{j}-\hat{k}\right)-\left(\hat{i}+2\hat{j}+\hat{k}\right)=\hat{i}-3\hat{j}-2\hat{k} \quad \text{.... (2)}$$

Now,

$$\overrightarrow{b_1}\times\overrightarrow{b_2}=\left(\hat{i}-\hat{j}+\hat{k}\right)\times\left(2\hat{i}+\hat{j}+2\hat{k}\right)$$

$$=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -1 & 1\\ 2 & 1 & 2\end{vmatrix}$$

$$=-3\hat{i}+3\hat{k}$$

$$\Rightarrow \overrightarrow{b_1}\times\overrightarrow{b_2}=-3\hat{i}+3\hat{k} \quad \text{.......... (3)}$$

$$\Rightarrow \left|\overrightarrow{b_1}\times\overrightarrow{b_2}\right|=\sqrt{(-3)^2+3^2}=\sqrt{9+9}=\sqrt{18}=3\sqrt{2} \quad \text{.......... (4)}$$

Now,

$$\left(a_1\hat{i}+b_1\hat{j}+c_1\hat{k}\right).\left(a_2\hat{i}+b_2\hat{j}+c_2\hat{k}\right)=a_1a_2+b_1b_2+c_1c_2$$

$$\left(\overrightarrow{b_1}\times\overrightarrow{b_2}\right).\left(\overrightarrow{a_2}-\overrightarrow{a_1}\right)=\left(-3\hat{i}+3\hat{k}\right).\left(\hat{i}-3\hat{j}-2\hat{k}\right)=-3-6=-9 \quad \text{.......... (5)}$$

Now, by substituting all the values in equation (1), we get

The shortest distance between the two lines,

$$d=\left|\frac{-9}{3\sqrt{2}}\right|$$

$$=\frac{9}{3\sqrt{2}} \text{ [From equation (4) and (5)]}$$

$$=\frac{3}{\sqrt{2}}$$

Let us rationalizing the fraction by multiplying the numerator and denominator by √2, we get

$$d=\frac{3}{\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}=\frac{3\sqrt{2}}{2}$$

∴ The shortest distance is $3\sqrt{2}/2$

15. Find the shortest distance between the lines

$$\frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1} \text{ and } \frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1}$$

Solution:

We know that the shortest distance between two lines

$$\frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1} \text{ and } \frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1} \text{ is given as:}$$

$$d=\frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2}} \quad \ldots (1)$$

The standard form of a pair of Cartesian lines is:

$$\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1} \text{ and } \frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$$

And the given equations are:

$$\frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1} \text{ and } \frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1}$$

Now let us compare the given equations with the standard form we get,

$x_1 = -1$, $y_1 = -1$, $z_1 = -1$;

$x_2 = 3$, $y_2 = 5$, $z_2 = 7$

$a_1 = 7$, $b_1 = -6$, $c_1 = 1$;

$a_2 = 1$, $b_2 = -2$, $c_2 = 1$

Now, consider

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \begin{vmatrix} 3-(-1) & 5-(-1) & 7-(-1) \\ 7 & -6 & 1 \\ 1 & -2 & 1 \end{vmatrix} = \begin{vmatrix} 3+1 & 5+1 & 7+1 \\ 7 & -6 & 1 \\ 1 & -2 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 4 & 6 & 8 \\ 7 & -6 & 1 \\ 1 & -2 & 1 \end{vmatrix}$$

= 4 (-6 + 2) – 6 (7 - 1) + 8 (-14 + 6)
= 4 (4) – 6 (6) + 8 (-8)
= -16 – 36 – 64
= - 116

Now we shall consider

$$\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}$$

$$= \sqrt{((-6\times1)-(-2\times1))^2 + ((1\times1)-(1\times7))^2 + ((7\times-2)-(1\times-6))^2}$$

$$= \sqrt{(-6+2)^2 + (1-7)^2 + (-14+6)^2} = \sqrt{(-4)^2 + (-6)^2 + (-8)^2}$$

$$= \sqrt{16+36+64} = \sqrt{116}$$

By substituting all the values in equation (1), we get
The shortest distance between the two lines,

$$d = \left|\frac{-116}{\sqrt{116}}\right| = \frac{116}{\sqrt{116}} = \sqrt{116} = 2\sqrt{29}$$

∴ The shortest distance is $2\sqrt{29}$

16. Find the shortest distance between the lines whose vector equations are

$$\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} - 3\hat{j} + 2\hat{k}) \text{ and}$$

$$\vec{r} = 4\hat{i} + 5\hat{j} - 6\hat{k} + \mu(2\hat{i} + 3\hat{j} + \hat{k})$$

Solution:

We know that shortest distance between two lines $\vec{r} = \vec{a_1} + \lambda\vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu\vec{b_2}$ is given as:

$$d = \left|\frac{(\vec{b_1} \times \vec{b_2}).(\vec{a_2} - \vec{a_1})}{|\vec{b_1} \times \vec{b_2}|}\right| \quad \text{.......... (1)}$$

Here by comparing the equations we get,

$\overrightarrow{a_1} = \hat{i} + 2\hat{j} + 3\hat{k}$, $\overrightarrow{b_1} = \hat{i} - 3\hat{j} + 2\hat{k}$ and

$\overrightarrow{a_2} = 4\hat{i} + 5\hat{j} + 6\hat{k}$, $\overrightarrow{b_2} = 2\hat{i} + 3\hat{j} + \hat{k}$

Now let us subtract the above equations we get,

$$\left(x_1\hat{i} + y_1\hat{j} + z_1\hat{k}\right) - \left(x_2\hat{i} + y_2\hat{j} + z_2\hat{k}\right) = (x_1 - x_2)\hat{i} + (y_1 - y_2)\hat{j} + (z_1 - z_2)\hat{k}$$

$$\overrightarrow{a_2} - \overrightarrow{a_1} = \left(4\hat{i} + 5\hat{j} + 6\hat{k}\right) - \left(\hat{i} + 2\hat{j} + 3\hat{k}\right) = 3\hat{i} + 3\hat{j} + 3\hat{k} \quad \text{.......... (2)}$$

And,

$$\overrightarrow{b_1} \times \overrightarrow{b_2} = \left(\hat{i} - 3\hat{j} + 2\hat{k}\right) \times \left(2\hat{i} + 3\hat{j} + \hat{k}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 2 \\ 2 & 3 & 1 \end{vmatrix}$$

$$= -9\hat{i} + 3\hat{j} + 9\hat{k}$$

$$\Rightarrow \overrightarrow{b_1} \times \overrightarrow{b_2} = -9\hat{i} + 3\hat{j} + 9\hat{k} \quad \text{.......... (3)}$$

$$\Rightarrow \left|\overrightarrow{b_1} \times \overrightarrow{b_2}\right| = \sqrt{(-9)^2 + 3^2 + 9^2} = \sqrt{81 + 9 + 81} = \sqrt{171} = 3\sqrt{19} \quad \text{..... (4)}$$

Now by multiplying equation (2) and (3) we get,

$$\left(a_1\hat{i} + b_1\hat{j} + c_1\hat{k}\right).\left(a_2\hat{i} + b_2\hat{j} + c_2\hat{k}\right) = a_1a_2 + b_1b_2 + c_1c_2$$

$$\left(\overrightarrow{b_1} \times \overrightarrow{b_2}\right).\left(\overrightarrow{a_2} - \overrightarrow{a_1}\right) = \left(-9\hat{i} + 3\hat{j} + 9\hat{k}\right).\left(3\hat{i} + 3\hat{j} + 3\hat{k}\right) = -27 + 9 + 27 = 9 \quad \text{.... (5)}$$

By substituting all the values in equation (1), we obtain

The shortest distance between the two lines,

$$d = \left|\frac{9}{3\sqrt{19}}\right| = \frac{9}{3\sqrt{19}} = \frac{3}{\sqrt{19}}$$

∴ The shortest distance is 3√19

17. Find the shortest distance between the lines whose vector equations are

$\vec{r} = (1-t)\hat{i} + (t-2)\hat{j} + (3-2t)\hat{k}$ and

$\vec{r} = (s+1)\hat{i} + (2s-1)\hat{j} - (2s+1)\hat{k}$

Solution:

Firstly let us consider the given equations

=> $\vec{r} = (1-t)\hat{i} + (t-2)\hat{j} + (3-2t)\hat{k}$

$\vec{r} = \hat{i} - t\hat{i} + t\hat{j} - 2\hat{j} + 3\hat{k} - 2t\hat{k}$

$\vec{r} = \hat{i} - 2\hat{j} + 3\hat{k} + t\left(-\hat{i} + \hat{j} - 2\hat{k}\right)$

=> $\vec{r} = (s+1)\hat{i} + (2s-1)\hat{j} - (2s+1)\hat{k}$

$\vec{r} = s\hat{i} + \hat{i} + 2s\hat{j} - \hat{j} - 2s\hat{k} - \hat{k}$

$\vec{r} = \hat{i} - \hat{j} - \hat{k} + s\left(\hat{i} + 2\hat{j} - 2\hat{k}\right)$

So now we need to find the shortest distance between

$\vec{r} = \hat{i} - 2\hat{j} + 3\hat{k} + t\left(-\hat{i} + \hat{j} - 2\hat{k}\right)$ and $\vec{r} = \hat{i} - \hat{j} - \hat{k} + s\left(\hat{i} + 2\hat{j} - 2\hat{k}\right)$

We know that shortest distance between two lines

$\vec{r} = \overrightarrow{a_1} + \lambda\overrightarrow{b_1}$ and $\vec{r} = \overrightarrow{a_2} + \mu\overrightarrow{b_2}$ is given as:

$$d = \left|\frac{\left(\overrightarrow{b_1} \times \overrightarrow{b_2}\right).\left(\overrightarrow{a_2} - \overrightarrow{a_1}\right)}{\left|\overrightarrow{b_1} \times \overrightarrow{b_2}\right|}\right| \quad \ldots (1)$$

Here by comparing the equations we get,

$\overrightarrow{a_1} = \hat{i} - 2\hat{j} + 3\hat{k}, \overrightarrow{b_1} = -\hat{i} + \hat{j} - 2\hat{k}$ and

$\overrightarrow{a_2} = \hat{i} - \hat{j} - \hat{k}, \overrightarrow{b_2} = \hat{i} + 2\hat{j} - 2\hat{k}$

Since,

$\left(x_1\hat{i} + y_1\hat{j} + z_1\hat{k}\right) - \left(x_2\hat{i} + y_2\hat{j} + z_2\hat{k}\right) = (x_1 - x_2)\hat{i} + (y_1 - y_2)\hat{j} + (z_1 - z_2)\hat{k}$

So,

$\overrightarrow{a_2} - \overrightarrow{a_1} = \left(\hat{i} - \hat{j} - \hat{k}\right) - \left(\hat{i} - 2\hat{j} + 3\hat{k}\right) = \hat{j} - 4\hat{k} \quad \ldots. (2)$

And,

$$\overrightarrow{b_1} \times \overrightarrow{b_2} = \left(-\hat{i} + \hat{j} - 2\hat{k}\right) \times \left(\hat{i} + 2\hat{j} - 2\hat{k}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & -2 \\ 1 & 2 & -2 \end{vmatrix}$$

$$= 2\hat{i} - 4\hat{j} - 3\hat{k}$$

$$\Rightarrow \overrightarrow{b_1} \times \overrightarrow{b_2} = 2\hat{i} - 4\hat{j} - 3\hat{k} \quad \ldots\ldots\ldots\ldots (3)$$

$$\Rightarrow \left|\overrightarrow{b_1} \times \overrightarrow{b_2}\right| = \sqrt{2^2 + (-4)^2 + (-3)^2} = \sqrt{4 + 16 + 9} = \sqrt{29} \quad \ldots\ldots\ldots\ldots (4)$$

Now by multiplying equation (2) and (3) we get,

$$\left(a_1\hat{i} + b_1\hat{j} + c_1\hat{k}\right).\left(a_2\hat{i} + b_2\hat{j} + c_2\hat{k}\right) = a_1a_2 + b_1b_2 + c_1c_2$$

$$\left(\overrightarrow{b_1} \times \overrightarrow{b_2}\right).\left(\overrightarrow{a_2} - \overrightarrow{a_1}\right) = \left(2\hat{i} - 4\hat{j} - 3\hat{k}\right).\left(\hat{j} - 4\hat{k}\right) = -4 + 12 = 8 \quad \ldots\ldots\ldots\ldots (5)$$

By substituting all the values in equation (1), we obtain
The shortest distance between the two lines,

$$d = \left|\frac{8}{\sqrt{29}}\right| = \frac{8}{\sqrt{29}}$$

∴ The shortest distance is 8√29

EXERCISE 11.3 PAGE NO: 493

1. In each of the following cases, determine the direction cosines of the normal to the plane and the distance from the origin.
(a) z = 2
(b) x + y + z = 1
(c) 2x + 3y – z = 5 (d) 5y + 8 = 0 Solution: (a) z = 2 Given:
The equation of the plane, z = 2 or 0x + 0y + z = 2 …. (1)
Direction ratio of the normal (0, 0, 1)
By using the formula,
$\sqrt{[(0)^2 + (0)^2 + (1)^2]} = \sqrt{1}$
$= 1$
Now,
Divide both the sides of equation (1) by 1, we get
0x/(1) + 0y/(1) + z/1 = 2
So this is of the form lx + my + nz = d
Where, l, m, n are the direction cosines and d is the distance ∴
The direction cosines are 0, 0, 1
Distance (d) from the origin is 2 units

(b) x + y + z = 1
Given:
The equation of the plane, x + y + z = 1…. (1)
Direction ratio of the normal (1, 1, 1)
By using the formula,
$\sqrt{[(1)^2 + (1)^2 + (1)^2]} = \sqrt{3}$
Now,
Divide both the sides of equation (1) by √3, we get
x/(√3) + y/(√3) + z/(√3) = 1/√3 So this is of the
form lx + my + nz = d
Where, l, m, n are the direction cosines and d is the distance
∴ The direction cosines are 1/√3, 1/√3, 1/√3

Distance (d) from the origin is 1/√3 units

(c) $2x + 3y – z = 5$
Given:
The equation of the plane, $2x + 3y – z = 5$…. (1)
Direction ratio of the normal (2, 3, -1)
By using the formula,
$\sqrt{[(2)^2 + (3)^2 + (-1)^2]} = \sqrt{14}$
Now,
Divide both the sides of equation (1) by √14, we get
$2x/(\sqrt{14}) + 3y/(\sqrt{14}) - z/(\sqrt{14}) = 5/\sqrt{14}$
So this is of the form $lx + my + nz = d$
Where, l, m, n are the direction cosines and d is the distance
∴ The direction cosines are 2/√14, 3/√14, -1/√14
Distance (d) from the origin is 5/√14 units

(d) $5y + 8 = 0$ Given:
The equation of the plane, $5y + 8 = 0$
$-5y = 8$ or
$0x – 5y + 0z = 8$…. (1)
Direction ratio of the normal (0, -5, 0)
By using the formula,
$\sqrt{[(0)^2 + (-5)^2 + (0)^2]} = \sqrt{25}$
$= 5$
Now,
Divide both the sides of equation (1) by 5, we get
$0x/(5) - 5y/(5) – 0z/(5) = 8/5$
So this is of the form $lx + my + nz = d$
Where, l, m, n are the direction cosines and d is the distance
∴ The direction cosines are 0, -1, 0
Distance (d) from the origin is 8/5 units

2. Find the vector equation of a plane which is at a distance of 7 units from the origin and normal to the vector

$3\hat{i}+5\hat{j}-6\hat{k}$.

Solution:

Given:

The vector $3\hat{i}+5\hat{j}-6\hat{k}$.

Vector eq. of the plane with position vector $\vec{r}$ is

$\vec{r}.\hat{n} = d$ … (1)

So,

$$\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{3\hat{i}+5\hat{j}-6\hat{k}}{\sqrt{9+25+36}}$$

$$= \frac{3\hat{i}+5\hat{j}-6\hat{k}}{\sqrt{70}}$$

Substituting in equation (1), we get

$$\vec{r}.\frac{3\hat{i}+5\hat{j}-6\hat{k}}{\sqrt{70}} = 7$$

$$\vec{r}.3\hat{i}+5\hat{j}-6\hat{k} = 7\sqrt{70}$$

∴ The required vector equation is $\vec{r}.3\hat{i}+5\hat{j}-6\hat{k} = 7\sqrt{70}$

3. Find the Cartesian equation of the following planes:

(a) $\vec{r}.(\hat{i}+\hat{j}-\hat{k}) = 2$

Solution:

Given:

The equation of the plane.

Let $\vec{r}$ be the position vector of P(x, y, z) is given by

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

So,

$$\vec{r}.\left(\hat{i}+\hat{j}-\hat{k}\right)=2$$

Substituting the value of $\vec{r}$, we get

$$\left(x\hat{i}+y\hat{j}+z\hat{k}\right).\left(\hat{i}+\hat{j}-\hat{k}\right)=2$$

∴ The Cartesian equation is

$x + y - z = 2$

(b) $\vec{r}.(2\hat{i} + 3\hat{j} - 4\hat{k}) = 1$

Solution:

Let $\vec{r}$ be the position vector of P(x, y, z) is given by

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

So,

$$\vec{r}.\left(2\hat{i}+3\hat{j}-\hat{4}k\right)=1$$

Substituting the value of $\vec{r}$, we get

$$\left(x\hat{i}+y\hat{j}+z\hat{k}\right).\left(2\hat{i}+3\hat{j}-4\hat{k}\right)=1$$

∴ The Cartesian equation is

$2x + 3y - 4z = 1$

(c) $\vec{r}.[(s-2t)\hat{i}+(3-t)\hat{j}+(2s+t)\hat{k}]=15$

Solution:

Let $\vec{r}$ be the position vector of P(x, y, z) is given by

$\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$

So,

$\vec{r}.\left[(s-2t)\hat{i}+(3-t)\hat{j}+(2s+t)\hat{k}\right]=15$

Substituting the value of $\vec{r}$, we get

$\left(x\hat{i}+y\hat{j}+z\hat{k}\right).\left[(s-2t)\hat{i}+(3-t)\hat{j}+(2s+t)\hat{k}\right]=15$

∴ The Cartesian equation is

$(s-2t)x+(3-t)y+(2s+t)z=15$

4. In the following cases, find the coordinates of the foot of the perpendicular drawn from the origin. (a) 2x + 3y + 4z – 12 = 0
(b) 3y + 4z – 6 = 0
(c) x + y + z = 1 (d) 5y + 8 = 0 Solution:

(a) 2x + 3y + 4z – 12 = 0

Let the coordinate of the foot of ⊥ P from the origin to the given plane be P(x, y, z).

2x + 3y + 4z = 12 …. (1)

Direction ratio are (2, 3, 4)

$\sqrt{[(2)^2 + (3)^2 + (4)^2]} = \sqrt{(4 + 9 + 16)}$
$= \sqrt{29}$

Now,

Divide both the sides of equation (1) by √29, we get

2x/(√29) + 3y/(√29) + 4z/(√29) = 12/√29

So this is of the form lx + my + nz = d

Where, l, m, n are the direction cosines and d is the distance

∴ The direction cosines are 2/√29, 3/√29, 4/√29

Coordinate of the foot (ld, md, nd) =
= [(2/√29) (12/√29), (3/√29) (12/√29), (4/√29) (12/√29)]
= 24/29, 36/29, 48/29

(b) 3y + 4z – 6 = 0

Let the coordinate of the foot of ⊥ P from the origin to the given plane be P(x, y, z).

$0x + 3y + 4z = 6$ …. (1)

Direction ratio are (0, 3, 4)

$\sqrt{[(0)^2 + (3)^2 + (4)^2]} = \sqrt{(0 + 9 + 16)}$

$= \sqrt{25}$

$= 5$

Now,

Divide both the sides of equation (1) by 5, we get

$0x/(5) + 3y/(5) + 4z/(5) = 6/5$

So this is of the form $lx + my + nz = d$

Where, l, m, n are the direction cosines and d is the distance

∴ The direction cosines are 0/5, 3/5, 4/5

Coordinate of the foot (ld, md, nd) =

$= [(0/5)(6/5), (3/5)(6/5), (4/5)(6/5)]$

$= 0, 18/25, 24/25$

(c) $x + y + z = 1$

Let the coordinate of the foot of ⊥ P from the origin to the given plane be P(x, y, z).

$x + y + z = 1$ …. (1) Direction
ratio are (1, 1, 1)

$\sqrt{[(1)^2 + (1)^2 + (1)^2]} = \sqrt{(1 + 1 + 1)}$

$= \sqrt{3}$

Now,

Divide both the sides of equation (1) by √3, we get

$1x/(\sqrt{3}) + 1y/(\sqrt{3}) + 1z/(\sqrt{3}) = 1/\sqrt{3}$

So this is of the form $lx + my + nz = d$

Where, l, m, n are the direction cosines and d is the distance

∴ The direction cosines are 1/√3, 1/√3, 1/√3

Coordinate of the foot (ld, md, nd) =

$= [(1/\sqrt{3})(1/\sqrt{3}), (1/\sqrt{3})(1/\sqrt{3}), (1/\sqrt{3})(1/\sqrt{3})]$

$= 1/3, 1/3, 1/3$

(d) $5y + 8 = 0$

Let the coordinate of the foot of ⊥ P from the origin to the given plane be P(x, y, z).

$0x - 5y + 0z = 8$ …. (1)

Direction ratio are (0, -5, 0)

$\sqrt{[(0)^2 + (-5)^2 + (0)^2]} = \sqrt{(0 + 25 + 0)}$

$= \sqrt{25}$

$= 5$

Now,

Divide both the sides of equation (1) by 5, we get

0x/(5) - 5y/(5) + 0z/(5) = 8/5

So this is of the form lx + my + nz = d

Where, l, m, n are the direction cosines and d is the distance

∴ The direction cosines are 0, -1, 0

Coordinate of the foot (ld, md, nd) =

= [(0/5) (8/5), (-5/5) (8/5), (0/5) (8/5)]

= 0, -8/5, 0

5. Find the vector and Cartesian equations of the planes

(a) that passes through the point (1, 0, –2) and the normal to the plane is

$\hat{i}+\hat{j}-\hat{k}$.

(b) that passes through the point (1,4, 6) and the normal vector to the plane is

$\hat{i}-2\hat{j}+\hat{k}$.

Solution:

(a) That passes through the point (1, 0, –2) and the normal to the plane is

$\hat{i}+\hat{j}-\hat{k}$.

Let the position vector of the point (1, 0, -2) be

$\vec{a} = \left(1\hat{i}+0\hat{j}-2\hat{k}\right)$

We know that Normal $\vec{N}$ ⊥ to the plane is given as:

$\vec{N} = \hat{i}+\hat{j}-\hat{k}$

Vector equation of the plane is given as:

$\left(\vec{r}-\vec{a}\right).\vec{N} = 0$

Now,

x – 1 - 2y + 8 + z – 6 = 0

x - 2y + z + 1 = 0 x

– 2y + z = -1

∴ The required Cartesian equation of the plane is x – 2y + z = -1

$$\left(\vec{r}-\left(\hat{i}-2\hat{k}\right)\right).\hat{i}+\hat{j}-\hat{k}=0 \quad \ldots (1)$$

Since,

$$\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$$

So equation (1) becomes,

$$\left(x\hat{i}+y\hat{j}+z\hat{k}-\hat{i}+2\hat{k}\right).\hat{i}+\hat{j}-\hat{k}=0$$

$$\left[(x-1)\hat{i}+y\hat{j}+(z+2)\hat{k}\right].\hat{i}+\hat{j}-\hat{k}=0$$

$$x-1+y-z-2=0$$

$$x+y-z-3=0$$

$\therefore$ The required Cartesian equation of the plane is $x+y-z=3$

(b) That passes through the point (1, 4, 6) and the normal vector to the plane is $\hat{i}-2\hat{j}+\hat{k}$.

Let the position vector of the point (1, 0, -2) be

$$\vec{a}=\left(1\hat{i}+4\hat{j}+6\hat{k}\right)$$

We know that Normal $\vec{N}$ $\perp$ to the plane is given as:

$$\vec{N}=\hat{i}-2\hat{j}+\hat{k}$$

Vector equation of the plane is given as:

$$(\vec{r}-\vec{a}).\vec{N}=0$$

Now,

$$\left(\vec{r}-\left(\hat{i}+4\hat{j}+6\hat{k}\right)\right).\hat{i}-2\hat{j}+\hat{k}=0 \quad \ldots (1)$$

Since,

$$\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$$

So equation (1) becomes,

$$\left(x\hat{i}+y\hat{j}+z\hat{k}-\hat{i}-4\hat{j}-6\hat{k}\right).\hat{i}-2\hat{j}+\hat{k}=0$$

$$\left[(x-1)\hat{i}+(y-4)\hat{j}+(z-6)\hat{k}\right].\hat{i}-2\hat{j}+\hat{k}=0$$

x – 1 - 2y + 8 + z – 6 = 0
x - 2y + z + 1 = 0 x – 2y
+ z = -1
∴ The required Cartesian equation of the plane is x – 2y + z = -1

6. Find the equations of the planes that passes through three points.
(a) (1, 1, –1), (6, 4, –5), (–4, –2, 3) (b) (1, 1, 0), (1, 2, 1), (–2, 2, –1) Solution:
Given:
The points are (1, 1, -1), (6, 4, -5), (-4, -2, 3).
Let,

$$=\begin{vmatrix} 1 & 1 & -1 \\ 6 & 4 & -5 \\ -4 & -2 & 3 \end{vmatrix}$$

= 1(12 - 10) – 1(18 - 20) -1 (-12 + 16)
= 2 + 2 – 4
= 0
Since, the value of determinant is 0.
∴ The points are collinear as there will be infinite planes passing through the given 3 points.

(b) (1, 1, 0), (1, 2, 1), (–2, 2, –1)

The given points are (1, 1, 0), (1, 2, 1), (-2, 2, -1).

Let,

$$=\begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ -2 & 2 & -1 \end{vmatrix}$$

= 1(-2 - 2) – 1(-1 + 2)

= -4 – 1

= -5 $\neq$ 0

Since, there passes a unique plane from the given 3 points.

Equation of the plane passing through the points, (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3), i.e.,

$$=\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}$$

Let us substitute the values and simplify

$$=\begin{vmatrix} x-1 & y-1 & z \\ x_2-1 & y_2-1 & z_2 \\ x_3-1 & y_3-1 & z_3 \end{vmatrix}$$

$$=\begin{vmatrix} x-1 & y-1 & z \\ 1-1 & 2-1 & 1 \\ -2-1 & 2-1 & -1 \end{vmatrix}$$

$$=\begin{vmatrix} x-1 & y-1 & z \\ 0 & 1 & 1 \\ -3 & 1 & -1 \end{vmatrix}$$

=> (x - 1) (-2) – (y - 1) (3) + 3z = 0

=> -2x + 2 – 3y + 3 + 3z = 0

= 2x + 3y – 3z = 5

∴ The required equation of the plane is 2x + 3y – 3z = 5.

7. Find the intercepts cut off by the plane 2x + y – z = 5.
Solution:
Given:
The plane 2x + y – z = 5
Let us express the equation of the plane in intercept form x/a
+ y/b + z/c = 1
Where a, b, c are the intercepts cut-off by the plane at x, y and z axes respectively.
2x + y – z = 5 …. (1)
Now divide both the sides of equation (1) by 5, we get
2x/5 + y/5 – z/5 = 5/5 2x/5
+ y/5 – z/5 = 1 x/(5/2) +
y/5 + z/(-5) = 1
Here, a = 5/2, b = 5 and c = -5
∴ The intercepts cut-off by the plane are 5/2, 5 and -5.

8. Find the equation of the plane with intercept 3 on the y-axis and parallel to ZOX plane. Solution:
We know that the equation of the plane ZOX is y = 0
So, the equation of plane parallel to ZOX is of the form, y = a
Since the y-intercept of the plane is 3, a = 3
∴ The required equation of the plane is y = 3

9. Find the equation of the plane through the intersection of the planes 3x – y + 2z – 4 = 0 and x + y + z – 2 = 0 and the point (2, 2, 1).
Solution:
Given:
Equation of the plane passes through the intersection of the plane is given by
(3x – y + 2z – 4) + λ (x + y + z – 2) = 0 and the plane passes through the points (2, 2, 1).
So, (3 × 2 – 2 + 2 × 1 – 4) + λ (2 + 2 + 1 – 2) = 0
2 + 3λ = 0 3λ
= -2 λ = -2/3
…. (1)
Upon simplification, the required equation of the plane is given as
(3x – y + 2z - 4) – 2/3 (x + y + z - 2) = 0

(9x – 3y + 6z – 12 – 2x – 2y – 2z + 4)/3 = 0
7x – 5y + 4z – 8 = 0
∴ The required equation of the plane is 7x – 5y + 4z – 8 = 0

10. Find the vector equation of the plane passing through the intersection of the planes $\vec{r}.(2\hat{i}+2\hat{j}-3\hat{k})=7,\ \vec{r}.(2\hat{i}+5\hat{j}+3\hat{k})=9$
and through the point (2, 1, 3). Solution:

Let the vector equation of the plane passing through the intersection of the planes are

$$\vec{r}.\left(2\hat{i}+2\hat{j}-3\hat{k}\right)=7 \text{ and } \vec{r}.\left(2\hat{i}+5\hat{j}+3\hat{k}\right)=9$$

Here,

$$\vec{r}.\left(2\hat{i}+2\hat{j}-3\hat{k}\right)-7=0 \quad \ldots. (1)$$

$$\vec{r}.\left(2\hat{i}+5\hat{j}+3\hat{k}\right)-9=0 \quad \ldots. (2)$$

The equation of any plane through the intersection of the planes given in equations (1) and (2) is given by,

$$\left[\vec{r}.\left(2\hat{i}+2\hat{j}-3\hat{k}\right)-7\right]+\lambda\left[\vec{r}.\left(2\hat{i}+5\hat{j}+3\hat{k}\right)-9\right]=0$$

$$\vec{r}\left[\left(2\hat{i}+2\hat{j}-3\hat{k}\right)+\left(2\lambda\hat{i}+5\lambda\hat{j}+3\lambda\hat{k}\right)\right]-7-9\lambda=0$$

$$\vec{r}.\left[(2+2\lambda)\hat{i}+(2+5\lambda)\hat{j}+(-3+3\lambda)\hat{k}\right]-7-9\lambda=0 \quad \ldots (3)$$

Since the plane passes through points (2, 1, 3)

$$\left(2\hat{i}+\hat{j}+3\hat{k}\right).\left[(2+2\lambda)\hat{i}+(2+5\lambda)\hat{j}+(-3+3\lambda)\hat{k}\right]-7-9\lambda=0$$

$4 + 4\lambda + 2 + 5\lambda - 9 + 9\lambda - 7 - 9\lambda = 0$

$9\lambda = 10$

$\lambda = 10/9$

Sow, substituting $\lambda = 10/9$ in equation (1) we get,

$$\vec{r}.\left[\left(2+\frac{20}{9}\right)\hat{i}+\left(2+\frac{50}{9}\right)\hat{j}+\left(-3+\frac{30}{9}\right)\hat{k}\right]-7-9\frac{10}{9}=0$$

$$\vec{r}.\left[\left(2+\frac{20}{9}\right)\hat{i}+\left(2+\frac{50}{9}\right)\hat{j}+\left(-3+\frac{30}{9}\right)\hat{k}\right]-17=0$$

$$\vec{r}.\left[\left(2+\frac{20}{9}\right)\hat{i}+\left(2+\frac{50}{9}\right)\hat{j}+\left(-3+\frac{30}{9}\right)\hat{k}\right]=17$$

$$\vec{r}\left[\frac{38}{9}\hat{i}+\frac{68}{9}\hat{j}+\frac{3}{9}\hat{k}\right]=17$$

$$\vec{r}\left[38\hat{i}+68\hat{j}+3\hat{k}\right]=153$$

$\therefore$ The required equation of the plane is $\vec{r}\left[38\hat{i}+68\hat{j}+3\hat{k}\right]=153$

11. Find the equation of the plane through the line of intersection of the planes x + y + z = 1 and 2x + 3y + 4z = 5 which is perpendicular to the plane x – y + z = 0.

Solution:

Let the equation of the plane that passes through the two-given planes
x + y + z = 1 and 2x + 3y + 4z = 5 is (x + y + z – 1) + λ (2x + 3y + 4z – 5) = 0

$(2\lambda + 1)$ x + $(3\lambda + 1)$ y + $(4\lambda + 1)$ z -1 - 5λ = 0…… (1)

So the direction ratio of the plane is $(2\lambda + 1, 3\lambda + 1, 4\lambda + 1)$

And direction ratio of another plane is (1, -1, 1)

Since, both the planes are ⊥

So by substituting in $a_1a_2 + b_1b_2 + c_1c_2 = 0$

$(2\lambda + 1 \times 1) + (3\lambda + 1 \times (-1)) + (4\lambda + 1 \times 1) = 0$

$2\lambda + 1 - 3\lambda - 1 + 4\lambda + 1 = 0$

$3\lambda + 1 = 0$ λ

= -1/3

Substitute the value of λ in equation (1) we get,

$$\left(2\frac{(-1)}{3}+1\right)x+\left(3\frac{(-1)}{3}+1\right)y+\left(4\frac{(-1)}{3}+1\right)z-1-5\frac{(-1)}{3}=0$$

$$\frac{1}{3}x-\frac{1}{3}z+\frac{2}{3}=0$$

$$x - z + 2 = 0$$

∴ The required equation of the plane is $x - z + 2 = 0$

12. Find the angle between the planes whose vector equations are

$\vec{r}.(2\hat{i}+2\hat{j}-3\hat{k})=5,\ \vec{r}.(3\hat{i}-3\hat{j}+5\hat{k})=3.$

Solution:

Given:

The equation of the given planes are

$$\vec{r}\left(2\hat{i}+2\hat{j}-3\hat{k}\right)=5 \text{ and } \vec{r}\left(3\hat{i}-3\hat{j}+5\hat{k}\right)=5$$

If n_1 and n_2 are normal to the planes, then

$$\vec{r_1}.\vec{n_1}=d_2 \text{ and } \vec{r_2}.\vec{n_2}=d_2$$

Angle between two planes is given as

$$\cos\theta=\left|\frac{\vec{n_1}.\vec{n_2}}{|\vec{n_1}||\vec{n_2}|}\right|$$

$$=\left|\frac{6-6-15}{\sqrt{4+4+9}\sqrt{9+9+25}}\right|$$

$$=\left|\frac{-15}{\sqrt{17}\sqrt{43}}\right|$$

$$\theta=\cos^{-1}\left(\frac{15}{\sqrt{17}\sqrt{43}}\right)$$

$$=\cos^{-1}\left(\frac{15}{\sqrt{731}}\right)$$

∴ The angle is $\cos^{-1}(15/\sqrt{731})$

13. In the following cases, determine whether the given planes are parallel or perpendicular, and in case they are neither, find the angles between them.

(a) 7x + 5y + 6z + 30 = 0 and 3x – y – 10z + 4 = 0

(b) 2x + y + 3z – 2 = 0 and x – 2y + 5 = 0

(c) 2x – 2y + 4z + 5 = 0 and 3x – 3y + 6z – 1 = 0

(d) 2x – 2y + 4z + 5 = 0 and 3x – 3y + 6z – 1 = 0 (e) 4x + 8y + z – 8 = 0 and y + z – 4 = 0 Solution:

(a) 7x + 5y + 6z + 30 = 0 and 3x – y – 10z + 4 = 0 Given:

The equation of the given planes are

7x + 5y + 6z + 30 = 0 and 3x – y – 10z + 4 = 0

Two planes are ⊥ if the direction ratio of the normal to the plane is

$a_1a_2 + b_1b_2 + c_1c_2 = 0$ 21
– 5 – 60
$-44 \neq 0$
Both the planes are not ⊥ to each other.
Now, two planes are || to each other if the direction ratio of the normal to the plane is

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

$$\frac{7}{3} \neq \frac{5}{-1} \neq \frac{6}{-10}$$ [both the planes are not || to each other]

Now, the angle between them is given by

$$\cos\theta = \left|\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}\right|$$

$$\cos\theta = \frac{-44}{\sqrt{49 + 25 + 36}\sqrt{9 + 1 + 100}}$$

$$= \frac{-44}{\sqrt{110}\sqrt{110}}$$

$$= \frac{-44}{110}$$

$$\theta = \cos^{-1}\frac{2}{5}$$

∴ The angle is $\cos^{-1}$ (2/5)

(b) 2x + y + 3z – 2 = 0 and x – 2y + 5 = 0 Given:
The equation of the given planes are
2x + y + 3z – 2 = 0 and x – 2y + 5 = 0
Two planes are ⊥ if the direction ratio of the normal to the plane is
$a_1a_2 + b_1b_2 + c_1c_2 = 0$ 2 × 1 + 1 × (-2) + 3 × 0
= 0
∴ The given planes are ⊥ to each other.

(c) $2x - 2y + 4z + 5 = 0$ and $3x - 3y + 6z - 1 = 0$
Given:
The equation of the given planes are
$2x - 2y + 4z + 5 = 0$ and $x - 2y + 5 = 0$
We know that, two planes are ⊥ if the direction ratio of the normal to the plane is
$a_1a_2 + b_1b_2 + c_1c_2 = 0$ $6 + 6 + 24$ $36 \neq 0$
∴ Both the planes are not ⊥ to each other.
Now let us check, both planes are || to each other if the direction ratio of the normal to the plane is

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

$$\frac{2}{3} = \frac{-2}{-3} = \frac{4}{6}$$

$$\frac{2}{3} = \frac{2}{3} = \frac{2}{3}$$

∴ The given planes are || to each other.

(d) $2x - 2y + 4z + 5 = 0$ and $3x - 3y + 6z - 1 = 0$ Given:
The equation of the given planes are
$2x - y + 3z - 1 = 0$ and $2x - y + 3z + 3 = 0$
We know that, two planes are ⊥ if the direction ratio of the normal to the plane is
$a_1a_2 + b_1b_2 + c_1c_2 = 0$ $2 \times 2 + (-1) \times (-1) + 3 \times 3$
$14 \neq 0$
∴ Both the planes are not ⊥ to each other.
Now, let us check two planes are || to each other if the direction ratio of the normal to the plane is

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

$$\frac{2}{2} = \frac{-1}{-1} = \frac{3}{3}$$

$$\frac{1}{1} = \frac{1}{1} = \frac{1}{1}$$

∴ The given planes are || to each other.

(e) $4x + 8y + z - 8 = 0$ and $y + z - 4 = 0$ Given:
The equation of the given planes are
$4x + 8y + z - 8 = 0$ and $y + z - 4 = 0$
We know that, two planes are ⊥ if the direction ratio of the normal to the plane is
$a_1a_2 + b_1b_2 + c_1c_2 = 0$ 0 + 8 + 1 9 ≠ 0
∴ Both the planes are not ⊥ to each other.
Now let us check, two planes are || to each other if the direction ratio of the normal to the plane is

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

$$\frac{4}{0} \neq \frac{8}{1} \neq \frac{1}{1}$$

∴ Both the planes are not || to each other.
Now let us find the angle between them which is given as

$$\cos\theta = \left| \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}} \right|$$

$$\cos\theta = \frac{4 \times 0 + 8 \times 1 + 1 \times 1}{\sqrt{16 + 64 + 1}\sqrt{0 + 1 + 1}}$$

$$= \frac{9}{9\sqrt{2}}$$

$$\theta = \cos^{-1}\frac{9}{9\sqrt{2}}$$

$$= \cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$= 45^0$$

∴ The angle is 45°.

14. In the following cases, find the distance of each of the given points from the corresponding given plane.

Point	**Plane**
(a) (0, 0, 0)	**3x – 4y + 12 z = 3**
(b) (3, -2, 1)	**2x – y + 2z + 3 = 0**
(c) (2, 3, -5)	**x + 2y – 2z = 9**
(d) (-6, 0, 0)	**2x – 3y + 6z – 2 = 0**

Solution:

(a) Point Plane

(0, 0, 0) 3x – 4y + 12 z = 3

We know that, distance of point P(x_1, y_1, z_1) from the plane Ax + By + Cz – D = 0 is given as:

$$d = \left|\frac{Ax_1 + By_1 + Cz_1 - D}{\sqrt{A^2 + B^2 + C^2}}\right|$$

Given point is (0, 0, 0) and the plane is 3x – 4y + 12z = 3

$$d = \left|\frac{0 + 0 + 0 + 3}{\sqrt{9 + 16 + 144}}\right|$$

= |3/√169|

= 3/13

∴ The distance is 3/13.

(b) Point Plane

(3, -2, 1) 2x – y + 2z + 3 = 0

We know that, distance of point P(x_1, y_1, z_1) from the plane Ax + By + Cz – D = 0 is given as:

$$d = \left|\frac{Ax_1 + By_1 + Cz_1 - D}{\sqrt{A^2 + B^2 + C^2}}\right|$$

Given point is (3, -2, 1) and the plane is 2x – y + 2z + 3 = 0

$$d = \left|\frac{6 + 2 + 2 + 3}{\sqrt{4 + 1 + 4}}\right|$$

= |13/√9|

= 13/3

∴ The distance is 13/3.

(c) Point Plane
(2, 3, -5) $x + 2y – 2z = 9$
We know that, distance of point $P(x_1, y_1, z_1)$ from the plane $Ax + By + Cz – D = 0$ is given as:

$$d = \left|\frac{Ax_1 + By_1 + Cz_1 - D}{\sqrt{A^2 + B^2 + C^2}}\right|$$

Given point is (2, 3, -5) and the plane is $x + 2y - 2z = 9$

$$d = \left|\frac{2 + 6 + 10 - 9}{\sqrt{1 + 4 + 4}}\right|$$

$= |9/\sqrt{9}|$
$= 9/3$
$= 3$
∴ The distance is 3.

(d) Point Plane
(-6, 0, 0) $2x – 3y + 6z – 2 = 0$
We know that, distance of point $P(x_1, y_1, z_1)$ from the plane $Ax + By + Cz – D = 0$ is given as:

$$d = \left|\frac{Ax_1 + By_1 + Cz_1 - D}{\sqrt{A^2 + B^2 + C^2}}\right|$$

Given point is (-6, 0, 0) and the plane is $2x - 3y + 6z - 2 = 0$

$$d = \left|\frac{-12 - 0 + 0 - 2}{\sqrt{4 + 9 + 36}}\right|$$

$= |14/\sqrt{49}|$
$= 14/7$
$= 2$
∴ The distance is 2.

MISCELLANEOUS EXERCISE PAGE NO: 497

1. Show that the line joining the origin to the point (2, 1, 1) is perpendicular to the line determined by the points (3, 5, –1), (4, 3, –1).

Solution:

Let us consider OA be the line joining the origin (0, 0, 0) and the point A (2, 1, 1).
And let BC be the line joining the points B (3, 5, −1) and C (4, 3, −1)
So the direction ratios of OA = $(a_1, b_1, c_1) \equiv [(2 - 0), (1 - 0), (1 - 0)] \equiv (2, 1, 1)$ And the direction ratios of BC = $(a_2, b_2, c_2) \equiv [(4 - 3), (3 - 5), (-1 + 1)] \equiv (1, -2, 0)$
Given:
OA is ⊥ to BC
Now we have to prove that:
$a_1a_2 + b_1b_2 + c_1c_2 = 0$
Let us consider LHS: $a_1a_2 + b_1b_2 + c_1c_2$
$a_1a_2 + b_1b_2 + c_1c_2 = 2 \times 1 + 1 \times (-2) + 1 \times 0$
$= 2 - 2$
$= 0$
We know that R.H.S is 0
So LHS = RHS
∴ OA is ⊥ to BC Hence proved.

2. If l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of two mutually perpendicular lines, show that the direction cosines of the line perpendicular to both of these are $(m_1n_2 - m_2n_1), (n_1l_2 - n_2l_1), (l_1m_2 - l_2m_1)$ Solution:

Let us consider l, m, n be the direction cosines of the line perpendicular to each of the given lines.

Then, $ll_1 + mm_1 + nn_1 = 0$ … (1)
And $ll_2 + mm_2 + nn_2 = 0$ … (2)
Upon solving (1) and (2) by using cross - multiplication, we get

$$\frac{l}{m_1n_2 - m_2n_1} = \frac{m}{n_1l_2 - n_2l_1} = \frac{n}{l_1m_2 - l_2m_1}$$

Thus, the direction cosines of the given line are proportional to
$(m_1n_2 - m_2n_1)$, $(n_1l_2 - n_2l_1)$, $(l_1m_2 - l_2m_1)$
So, its direction cosines are

$$\frac{m_1n_2 - m_2n_1}{\lambda}, \frac{n_1l_2 - n_2l_1}{\lambda}, \frac{l_1m_2 - l_2m_1}{\lambda}$$

Where,

$$\lambda = \sqrt{(m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2}$$

We know that
$(l_1^2 + m_1^2 + n_1^2)(l_2^2 + m_2^2 + n_2^2) - (l_1l_2 + m_1m_2 + n_1n_2)^2$
$= (m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2$ … (3) It is given that the given lines are perpendicular to each other.
So, $l_1l_2 + m_1m_2 + n_1n_2 = 0$
Also, we have $l_1^2 + m_1^2 + n_1^2 = 1$ And, $l_2^2 + m_2^2 + n_2^2 = 1$
Substituting these values in equation (3), we get
$(m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2 = 1$ $\lambda = 1$
Hence, the direction cosines of the given line are $(m_1n_2 - m_2n_1)$, $(n_1l_2 - n_2l_1)$, $(l_1m_2 - l_2m_1)$

3. Find the angle between the lines whose direction ratios are a, b, c and b – c, c – a, a – b.

Solution:
Angle between the lines with direction ratios a_1, b_1, c_1 and a_2, b_2, c_2 is given by

$$\cos\theta = \left|\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}\right|$$

Given:

$a_1 = a$, $b_1 = b$, $c_1 = c$
$a_2 = b - c$, $b_2 = c - a$, $c_2 = a - b$
Let us substitute the values in the above equation we get,

$$\cos\theta = \left|\frac{a(b-c)+b(c-a)+c(a-b)}{\sqrt{a^2+b^2+c^2}\sqrt{(b-c)^2+(c-a)^2+(a-b)^2}}\right|$$

$= 0$
Cos $\theta = 0$
So, $\theta = 90°$ [Since, cos 90 = 0]
Hence, Angle between the given pair of lines is 90°.

4. Find the equation of a line parallel to x - axis and passing through the origin.
Solution:
We know that, equation of a line passing through (x_1, y_1, z_1) and parallel to a line with direction ratios a, b, c is

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$$

Given: the line passes through origin i.e. (0, 0, 0) x_1
$= 0$, $y_1 = 0$, $z_1 = 0$
Since line is parallel to x - axis, a
$= 1$, $b = 0$, $c = 0$
∴ Equation of Line is given by

$$\frac{x-0}{1} = \frac{y-0}{0} = \frac{z-0}{0}$$

$$\frac{x}{1} = \frac{y}{0} = 0$$

5. If the coordinates of the points A, B, C, D be (1, 2, 3), (4, 5, 7), (–4, 3, –6) and (2, 9, 2) respectively, then find the angle between the lines AB and CD.
Solution:
We know that the angle between the lines with direction ratios a_1, b_1, c_1 and a_2, b_2, c_2 is given by

$$\cos\theta = \left|\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}\right|$$

So now, a line passing through A (x_1, y_1, z_1) and B (x_2, y_2, z_2) has direction ratios (x_1 - x_2), (y_1 - y_2), (z_1 - z_2)

The direction ratios of line joining the points A (1, 2, 3) and B (4, 5, 7)

= (4 - 1), (5 - 2), (7 - 3)

= (3, 3, 4) ∴

$a_1 = 3$, $b_1 = 3$, $c_1 = 4$

The direction ratios of line joining the points C (-4, 3, -6) and B (2, 9, 2)

= (2 - (-4)), (9 - 3), (2-(-6))

= (6, 6, 8) ∴

$a_2 = 6$, $b_2 = 6$, $c_2 = 8$

Now let us substitute the values in the above equation we get,

$$\cos\theta = \left|\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}\right|$$

$$\cos\theta = \left|\frac{3\times6 + 3\times6 + 4\times8}{\sqrt{3^2 + 3^2 + 4^2}\sqrt{6^2 + 6^2 + 8^2}}\right|$$

$$= \left|\frac{18+18+32}{\sqrt{9+9+16}\sqrt{36+36+64}}\right|$$

$$= \left|\frac{68}{\sqrt{34}\sqrt{136}}\right|$$

$$= \left|\frac{68}{\sqrt{34}\sqrt{4\times34}}\right|$$

$$= \left|\frac{68}{34\times2}\right|$$

$\cos\theta = 1$

So, $\theta = 0°$ [since, cos 0 is 1]

Hence, Angle between the lines AB and CD is 0°.

6. If the lines $\frac{x-1}{3k}=\frac{y-2}{1}=\frac{z-3}{-5}$ **and** $\frac{x-1}{3k}=\frac{y-2}{1}=\frac{z-3}{-5}$ **are perpendicular, find the value of k.**

Solution:

We know that the two lines

$\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$ which are perpendicular to each other if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

It is given that:

$$\frac{x-1}{-3}=\frac{y-2}{2k}=\frac{z-3}{2}$$

Let us compare with

$$\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$$

We get -

$x_1 =1, y_1 = 2, z_1 = 3$

And $a_1 = -3, b_1 = 2k, c_1 = 2$

Similarly,

We have,

$$\frac{x-1}{3k}=\frac{y-2}{1}=\frac{z-3}{-5}$$

Let us compare with

$$\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$$

We get -

$x_2 = 1, y_2 = 2, z_2 = 3$ And

$a_2 = 3k, b_2 = 1, c_2 = -5$

Since the two lines are perpendicular, a_1a_2
$+ b_1b_2 + c_1c_2 = 0$
$(-3) \times 3k + 2k \times 1 + 2 \times (-5) = 0$
$-9k + 2k - 10 = 0$
$-7k = 10$ $k = -$
10/7
7
∴ The value of k is -10/7.

7. Find the vector equation of the line passing through (1, 2, 3) and perpendicular to the plane

$$\vec{r}.\left(\hat{i}+2\hat{j}-5\hat{k}\right)+9=0$$

Solution:

The vector equation of a line passing through a point with position vector $\vec{a}$ and parallel to vector $\vec{b}$ is given as

$$\vec{r} = \vec{a} + \lambda \vec{b}$$

It is given that the line passes through (1, 2, 3)

So, $\vec{a} = 1\hat{i} + 2\hat{j} + 3\hat{k}$

Let us find the normal of plane

$$\vec{r}.\left(\hat{i} + 2\hat{j} - 5\hat{k}\right) + 9 = 0$$

$$\vec{r}.\left(\hat{i} + 2\hat{j} - 5\hat{k}\right) = -9$$

$$-\vec{r}.\left(\hat{i} + 2\hat{j} - 5\hat{k}\right) = 9$$

$$\vec{r}.\left(-1\hat{i} - 2\hat{j} + 5\hat{k}\right) + 9 = 0$$

Now compare it with $\vec{r}.\vec{n} = d$

$\vec{n} = -\hat{i} - 2\hat{j} + 5\hat{k}$

Since line is perpendicular to plane, the line will be parallel of the plane

$\therefore\ \vec{b} = \vec{n} = -\hat{i} - 2\hat{j} + 5\hat{k}$

Hence,

$$\vec{r} = \left(1\hat{i} + 2\hat{j} + 3\hat{k}\right) + \lambda\left(-\hat{i} - 2\hat{j} + 5\hat{k}\right)$$

$$\vec{r} = \left(1\hat{i} + 2\hat{j} + 3\hat{k}\right) - \lambda\left(\hat{i} + 2\hat{j} - 5\hat{k}\right)$$

$\therefore$ The required vector equation of line is $\vec{r} = \left(1\hat{i} + 2\hat{j} + 3\hat{k}\right) - \lambda\left(\hat{i} + 2\hat{j} - 5\hat{k}\right)$

8. Find the equation of the plane passing through (a, b, c) and parallel to the plane $\vec{r}.\left(\hat{i} + \hat{j} + \hat{k}\right) = 2$

Solution:

The equation of a plane passing through (x_1, y_1, z_1) and perpendicular to a line with direction ratios A, B, C is given as $A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$

It is given that, the plane passes through (a, b, c)

So, $x_1 = a$, $y_1 = b$, $z_1 = c$

Since both planes are parallel to each other, their normal will be parallel

$\therefore$ Direction ratios of normal of $\vec{r}.\left(\hat{i} + \hat{j} + \hat{k}\right) = 2$

Direction ratios of normal = (1, 1, 1)

So, A = 1, B =1, C = 1

The Equation of plane in Cartesian form is given as

$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$
$1(x - a) + 1(y - b) + 1(z - c) = 0$
$x + y + z - (a + b + c) = 0$ $x + y$
$+ z = a + b + c$
$\therefore$ The required equation of plane is $x + y + z = a + b + c$

9. Find the shortest distance between lines
$\vec{r} = \left(6\hat{i} + 2\hat{j} + 2\hat{k}\right) + \lambda\left(1\hat{i} - 2\hat{j} + \hat{2}\right)$ **and** $\vec{r} = \left(-4\hat{i} - \hat{k}\right) + \mu\left(3\hat{i} - 2\hat{j} - 2\hat{k}\right)$

Solution:

We know that the shortest distance between lines with vector equations $\vec{r} = \overrightarrow{a_1} + \lambda \overrightarrow{b_1}$ and $\vec{r} = \overrightarrow{a_2} + \lambda \overrightarrow{b_2}$ is given as

$$\left| \frac{\left(\overrightarrow{b_1} \times \overrightarrow{b_2}\right).\left(\overrightarrow{a_2} - \overrightarrow{a_1}\right)}{\left|\overrightarrow{b_1} \times \overrightarrow{b_2}\right|} \right|$$

It is given that:

$$\vec{r} = \left(6\hat{i} + 2\hat{j} + 2\hat{k}\right) + \lambda\left(1\hat{i} - 2\hat{j} + \hat{2}\right)$$

Now let us compare it with $\vec{r} = \overrightarrow{a_1} + \lambda \overrightarrow{b_1}$, we get

$\overrightarrow{a_1} = \left(6\hat{i} + 2\hat{j} + 2\hat{k}\right)$ and $\overrightarrow{b_1} = \left(1\hat{i} - 2\hat{j} + \hat{2}\right)$

Similarly,

$$\vec{r} = \left(-4\hat{i} - \hat{k}\right) + \mu\left(3\hat{i} - 2\hat{j} - 2\hat{k}\right)$$

Let us compare it with $\vec{r} = \overrightarrow{a_2} + \lambda \overrightarrow{b_2}$, we get

$\overrightarrow{a_2} = \left(-4\hat{i} - \hat{k}\right)$ and $\overrightarrow{b_2} = \left(3\hat{i} - 2\hat{j} - \hat{2}\right)$

Now,

$$\begin{aligned}\left(\overrightarrow{a_2} - \overrightarrow{a_1}\right) &= \left(-4\hat{i} - \hat{k}\right) - \left(6\hat{i} + 2\hat{j} + 2\hat{k}\right) \\ &= \left((-4-6)\hat{i} + (0-2)\hat{j} + (-1-2)\hat{k}\right) \\ &= \left(-10\hat{i} - 2\hat{j} - 3\hat{k}\right)\end{aligned}$$

And,

$$\left(\overrightarrow{b_1}\times\overrightarrow{b_2}\right)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -2 & 2\\ 3 & -2 & -2\end{vmatrix}$$

$$=\hat{i}\left[(-2\times-2)-(-2\times2)\right]-\hat{j}\left[(1\times-2)-(3\times2)\right]+\hat{k}\left[(1\times-2)-(3\times-2)\right]$$

$$=\hat{i}[4+4]-\hat{j}[-2-6]+\hat{k}[-2+6]$$

$$=8\hat{i}+8\hat{j}+4\hat{k}$$

So, Magnitude of $\overrightarrow{b_1}\times\overrightarrow{b_2}=\left|\overrightarrow{b_1}\times\overrightarrow{b_2}\right|=\sqrt{8^2+8^2+4^2}=\sqrt{64+64+16}$

$= \sqrt{144}$

$= 12$

Also,

$$\left(\overrightarrow{b_1}\times\overrightarrow{b_2}\right).\left(\overrightarrow{a_2}-\overrightarrow{a_1}\right)=\left(8\hat{i}+8\hat{j}+4\hat{k}\right).\left(-10\hat{i}-2\hat{j}-3\hat{k}\right)$$

$= -80 + (-16) + (-12)$

$= -108$

Hence the shortest distance is given as

$$=\left|\frac{\left(\overrightarrow{b_1}\times\overrightarrow{b_2}\right).\left(\overrightarrow{a_2}-\overrightarrow{a_1}\right)}{\left|\overrightarrow{b_1}\times\overrightarrow{b_2}\right|}\right|=\left|\frac{-108}{12}\right|=|-9|$$

$= 9$

∴ The shortest distance between the given two lines is 9.

10. Find the coordinates of the point where the line through (5, 1, 6) and (3, 4,1) crosses the YZ - plane. Solution:

We know that the vector equation of a line passing through two points with position vectors $\vec{a}$ and $\vec{b}$ is given as

$$\vec{r}=\vec{a}+\lambda\left(\vec{b}-\vec{a}\right)$$

So the position vector of point A (5, 1, 6) is given as

$\vec{a}=5\hat{i}+\hat{j}+6\hat{k}$ …. (1)

And the position vector of point B (3, 4, 1) is given as

$\vec{b} = 3\hat{i} + 4\hat{j} + \hat{k}$ … (2)

So subtract equation (2) and (1) we get

$$\left(\vec{b} - \vec{a}\right) = \left(3\hat{i} + 4\hat{j} + \hat{k}\right) - \left(5\hat{i} + \hat{j} + 6\hat{k}\right)$$
$$= (3-5)\hat{i} + (4-1)\hat{j} + (1-6)\hat{k}$$
$$= \left(-2\hat{i} + 3\hat{j} - 5\hat{k}\right)$$

$$\vec{r} = \left(5\hat{i} + \hat{j} + 6\hat{k}\right) + \lambda\left(-2\hat{i} + 3\hat{j} - 5\hat{k}\right) \text{ ... (3)}$$

Let the coordinates of the point where the line crosses the YZ plane be (0, y, z)

So,

$$\vec{r} = \left(0\hat{i} + y\hat{j} + z\hat{k}\right) \text{ .... (4)}$$

Since the point lies in line, it satisfies its equation,

Now substituting equation (4) in equation (3) we get,

$$\left(0\hat{i} + y\hat{j} + z\hat{k}\right) = \left(5\hat{i} + \hat{j} + 6\hat{k}\right) + \lambda\left(-2\hat{i} + 3\hat{j} - 5\hat{k}\right)$$
$$= (5 - 2\lambda)\hat{i} + (1 + 3\lambda)\hat{j} + (6 - 5\lambda)\hat{k}$$

We know that, two vectors are equal if their corresponding components are equal So,

$0 = 5 - 2\lambda$

$5 = 2\lambda$ λ

$= 5/2$

$y = 1 + 3\lambda$ … (5) And,

$z = 6 - 5\lambda$ … (6)

Substitute the value of λ in equation (5) and (6), we get -

$y = 1 + 3\lambda$ $= 1 + 3 \times (5/2)$

$= 1 + (15/2)$

$= 17/2$

And

$z = 6 - 5\lambda$ $= 6$

$- 5 \times (5/2)$

$= 6 - (25/2)$

$= - 13/2$

∴ The coordinates of the required point is (0, 17/2, -13/2).

11. Find the coordinates of the point where the line through (5, 1, 6) and (3, 4, 1) crosses the ZX - plane. Solution:

We know that the vector equation of a line passing through two points with position vectors $\vec{a}$ and $\vec{b}$ is given as

$$\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$$

So the position vector of point A (5, 1, 6) is given as

$$\vec{a} = 5\hat{i} + \hat{j} + 6\hat{k} \quad \ldots. (1)$$

And the position vector of point B (3, 4, 1) is given as

$$\vec{b} = 3\hat{i} + 4\hat{j} + \hat{k} \quad \ldots (2)$$

So subtract equation (2) and (1) we get

$$(\vec{b} - \vec{a}) = (3\hat{i} + 4\hat{j} + \hat{k}) - (5\hat{i} + \hat{j} + 6\hat{k})$$

$$= (3-5)\hat{i} + (4-1)\hat{j} + (1-6)\hat{k}$$

$$= (-2\hat{i} + 3\hat{j} - 5\hat{k})$$

$$\vec{r} = (5\hat{i} + \hat{j} + 6\hat{k}) + \lambda(-2\hat{i} + 3\hat{j} - 5\hat{k}) \quad \ldots (3)$$

Let the coordinates of the point where the line crosses the ZX plane be (0, y)

So,

$$\vec{r} = (x\hat{i} + 0\hat{j} + z\hat{k}) \quad \ldots. (4)$$

Since the point lies in line, it satisfies its equation,

Now substituting equation (4) in equation (3) we get,

$$(x\hat{i} + 0\hat{j} + z\hat{k}) = (5\hat{i} + \hat{j} + 6\hat{k}) + \lambda(-2\hat{i} + 3\hat{j} - 5\hat{k})$$

$$= (5 - 2\lambda)\hat{i} + (1 + 3\lambda)\hat{j} + (6 - 5\lambda)\hat{k}$$

We know that, two vectors are equal if their corresponding components are equal

So,

x = 5 - 2λ … (5)

$0 = 1 + 3\lambda$
$-1 = 3\lambda$
$\lambda = -1/3$ And,
$z = 6 - 5\lambda$ … (6)
Substitute the value of λ in equation (5) and (6), we get -
$x = 5 - 2\lambda \quad = 5 - 2 \times (-1/3)$
$= 5 + (2/3)$
$= 17/3$
And
$z = 6 - 5\lambda \quad = 6$
$- 5 \times (-1/3)$
$= 6 + (5/3)$
$= 23/3$
∴ The coordinates of the required point is (17/3, 0, 23/3).

12. Find the coordinates of the point where the line through (3, –4, –5) and (2, –3, 1) crosses the plane 2x + y + z = 7.
Solution:
We know that the equation of a line passing through two points A (x_1, y_1, z_1) and B (x_2, y_2, z_2) is given as

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$

It is given that the line passes through the points A (3, –4, –5) and B (2, –3, 1)
So, $x_1 = 3$, $y_1 = -4$, $z_1 = -5$
And, $x_2 = 2$, $y_2 = -3$, $z_2 = 1$
Then the equation of line is

$$\frac{x-3}{2-3}=\frac{y-(-4)}{-3-(-4)}=\frac{z-(-5)}{1-(-5)}$$

$$\frac{x-3}{-1}=\frac{y+4}{1}=\frac{z+5}{6}=k$$

So, $x = -k + 3$ |, $y = k - 4$ |, $z = 6k - 5$ … (1)
Now let (x, y, z) be the coordinates of the point where the line crosses the given plane $2x + y + z + 7 = 0$

By substituting the value of x, y, z in equation (1) in the equation of plane, we get
$2x + y + z + 7 = 0$
$2(-k + 3) + (k - 4) + (6k - 5) = 7$
$5k - 3 = 7$
$5k = 10$ k
$= 2$
Now substitute the value of k in x, y, z we get,
$x = - k + 3 = - 2 + 3 = 1$ $y = k - 4 = 2 - 4 = - 2$
$z = 6k - 5 = 12 - 5 = 7$
∴ The coordinates of the required point are (1, -2, 7).

13. Find the equation of the plane passing through the point (–1, 3, 2) and perpendicular to each of the planes x + 2y + 3z = 5 and 3x + 3y + z = 0.
Solution:
We know that the equation of a plane passing through (x_1, y_1, z_1) is given by
$A (x - x_1) + B (y - y_1) + C (z - z_1) = 0$
Where, A, B, C are the direction ratios of normal to the plane.
It is given that the plane passes through (-1, 3, 2)
So, equation of plane is given by
$A (x + 1) + B (y - 3) + C (z - 2) = 0$ ……… (1)
Since this plane is perpendicular to the given two planes. So, their normal to the plane would be perpendicular to normal of both planes.
We know that
$\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$
So, required normal is cross product of normal of planes x + 2y + 3z = 5 and 3x + 3y + z = 0

$$\text{Required Normal} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 3 & 3 & 1 \end{vmatrix}$$

$$= \hat{i}[2(1)-3(3)] - \hat{j}[1(1)-3(3)] + \hat{k}[1(3)-3(2)]$$

$$= \hat{i}[2-9] - \hat{j}[1-9] + \hat{k}[3-6]$$

$$= -7\hat{i} + 8\hat{j} - 3\hat{k}$$

Hence, the direction ratios are = -7, 8, -3

∴ A = -7, B = 8, C = -3

Substituting the obtained values in equation (1), we get

A (x + 1) + B (y - 3) + C (z - 2) = 0

-7(x + 1) + 8(y - 3) + (-3) (z - 2) = 0

-7x - 7 + 8y - 24 - 3z + 6 = 0

-7x + 8y - 3z - 25 = 0

7x - 8y + 3z + 25 = 0

∴ The equation of the required plane is 7x - 8y + 3z + 25 = 0.

14. If the points (1, 1, p) and (–3, 0, 1) be equidistant from the plane $\vec{r}.\left(3\hat{i}+4\hat{j}-12\hat{k}\right)+13=0$ **, then find the value of p.**

Solution:

We know that the distance of a point with position vector $\vec{a}$ from the plane $\vec{r}.\vec{n} = d$ is given as

$$\left|\frac{\vec{a}.\vec{n} - d}{|\vec{n}|}\right|$$

Now, the position vector of point (1, 1, p) is given as

$$\overrightarrow{a_1} = 1\hat{i} + 1\hat{j} + p\hat{k}$$

And, the position vector of point (-3, 0, 1) is given as

$$\overrightarrow{a_2} = -3\hat{i} + 0\hat{j} + 1\hat{k}$$

It is given that the points (1, 1, p) and (-3, 0, 1) are equidistant from the plane

$$\vec{r}.\left(3\hat{i} + 4\hat{j} - 12\hat{k}\right) + 13 = 0$$

So,

$$\left|\frac{\left(1\hat{i}+1\hat{j}+p\hat{k}\right).\left(3\hat{i}+4\hat{j}-12\hat{k}\right)+13}{\sqrt{3^2+4^2+(-12)^2}}\right| = \left|\frac{\left(-3\hat{i}+0\hat{j}+1\hat{k}\right).\left(3\hat{i}+4\hat{j}-12\hat{k}\right)+13}{\sqrt{3^2+4^2+(-12)^2}}\right|$$

$$\left|\frac{3+4-12p+13}{\sqrt{9+16+144}}\right| = \left|\frac{-9+0-12+13}{\sqrt{9+16+144}}\right|$$

$$\left|\frac{20-12p}{\sqrt{169}}\right| = \left|\frac{-8}{\sqrt{169}}\right|$$

$$|20-12p| = 8$$

20 - 12p = ± 8
20 - 12p = 8 or, 20 - 12p = -8
12p = 12 or, 12p = 28
p = 1 or, p = 7/3
∴ The possible values of p are 1 and 7/3.

15. Find the equation of the plane passing through the line of intersection of the planes $\vec{r}.\left(\hat{i}+\hat{j}+\hat{k}\right) = 1$ **and** $\vec{r}.\left(2\hat{i}+3\hat{j}-\hat{k}\right)+4=0$ **and parallel to x-axis.**

Solution:

We know that,

The equation of any plane through the line of intersection of the planes $\vec{r}.\overrightarrow{n_1} = d_1$ and $\vec{r}.\overrightarrow{n_2} = d_2$ is given by $\left(\vec{r}.\overrightarrow{n_1} - d_1\right) + \lambda\left(\vec{r}.\overrightarrow{n_2} - d_2\right) = 0$

So, the equation of any plane through the line of intersection of the given planes is

$$\left[\vec{r}.\left(\hat{i}+\hat{j}+\hat{k}\right)-1\right]+\lambda\left[\vec{r}.\left(-2\hat{i}-3\hat{j}+\hat{k}\right)-4\right]=0$$

$$\vec{r}.\left((1-2\lambda)\hat{i}+(1-3\lambda)\hat{j}+(1+\lambda)\hat{k}\right)-1-4\lambda=0$$

$$\vec{r}.\left((1-2\lambda)\hat{i}+(1-3\lambda)\hat{j}+(1+\lambda)\hat{k}\right)=1+4\lambda \quad \ldots (1)$$

Since this plane is parallel to x-axis.

So, the normal vector of the plane (1) will be perpendicular to x-axis. The direction ratios of Normal $(a_1, b_1, c_1) \equiv [(1 - 2\lambda), (1 - 3\lambda), (1 +)]$

The direction ratios of x–axis $(a_2, b_2, c_2) \equiv (1, 0, 0)$

Since the two lines are perpendicular, a_1a_2
$+ b_1b_2 + c_1c_2 = 0$

$(1 - 2\lambda) \times 1 + (1 - 3\lambda) \times 0 + (1 + \lambda) \times 0 = 0$

$(1 - 2\lambda) = 0$ λ
$= 1/2$

Substituting the value of λ in equation (1), we get

$$\vec{r}.\left((1-2\lambda)\hat{i}+(1-3\lambda)\hat{j}+(1+\lambda)\hat{k}\right)=1+4\lambda$$

$$\vec{r}.\left(\left(1-2\left(\frac{1}{2}\right)\right)\hat{i}+\left(1-3\left(\frac{1}{2}\right)\right)\hat{j}+\left(1+\frac{1}{2}\right)\hat{k}\right)=1+4\left(\frac{1}{2}\right)$$

$$\vec{r}.\left(0\hat{i}-\hat{j}+3\hat{k}\right)=6$$

$\therefore$ The equation of the required plane is $\vec{r}.\left(0\hat{i}-\hat{j}+3\hat{k}\right)=6$

16. If O be the origin and the coordinates of P be (1, 2, –3), then find the equation of the plane passing through P and perpendicular to OP.

Solution:

We know that the equation of a plane passing through (x_1, y_1, z_1) and perpendicular to a line with direction ratios A, B, C is given as

$A(x - x_1) + B(y - y_1) + C (z - z_1) = 0$

It is given that the plane passes through P (1, 2, 3)

So, $x_1 = 1, y_1 = 2, z_1 = - 3$

Normal vector to plane is = $\overrightarrow{OP}$

Where O (0, 0, 0), P (1, 2, -3)

So, direction ratios of $\overrightarrow{OP}$ is = (1 - 0), (2 - 0), (-3 - 0)

= (1, 2, - 3)

Where, A = 1, B = 2, C = -3

Equation of plane in Cartesian form is given as

$1(x - 1) + 2(y - 2) - 3(z - (-3)) = 0$

$x - 1 + 2y - 4 - 3z - 9 = 0$ $x + 2y -$

$3z - 14 = 0$

∴ The equation of the required plane is $x + 2y - 3z - 14 = 0$

17. Find the equation of the plane which contains the line of intersection of the planes $\vec{r}.(\hat{i}+2\hat{j}+3\hat{k})-4=0$ and $\vec{r}.(2\hat{i}+\hat{j}-\hat{k})+5=0$
And which is perpendicular to the plane $\vec{r}.(5\hat{i}+3\hat{j}-6\hat{k})+8=0$

Solution:

We know,

The equation of any plane through the line of intersection of the planes $\vec{r}.\overrightarrow{n_1}=d_1$ and $\vec{r}.\overrightarrow{n_2}=d_2$ is given by $(\vec{r}.\overrightarrow{n_1}-d_1)+\lambda(\vec{r}.\overrightarrow{n_2}-d_2)=0$

So, the equation of any plane through the line of intersection of the given planes is

$$\left[\vec{r}.(\hat{i}+2\hat{j}+3\hat{k})-4\right]+\lambda\left[\vec{r}.(-2\hat{i}-\hat{j}+\hat{k})-5\right]=0$$

$$\vec{r}.\left((1-2\lambda)\hat{i}+(2-\lambda)\hat{j}+(3+\lambda)\hat{k}\right)-4-5\lambda=0$$

$$\vec{r}.\left((1-2\lambda)\hat{i}+(2-\lambda)\hat{j}+(3+\lambda)\hat{k}\right)=4+5\lambda \quad \dots (1)$$

Since this plane is perpendicular to the plane

$$\bar{r}.\left(5\hat{i}+3\hat{j}-6\hat{k}\right)+8=0$$

$$\bar{r}.\left(5\hat{i}+3\hat{j}-6\hat{k}\right)=-8$$

$$-\bar{r}.\left(5\hat{i}+3\hat{j}-6\hat{k}\right)=8$$

$$\bar{r}.\left(-5\hat{i}-3\hat{j}+6\hat{k}\right)=8 \quad \ldots (2)$$

So, the normal vector of the plane (1) will be perpendicular to the normal vector of plane (2).

Direction ratios of Normal of plane (1) = (a_1, b_1, c_1) ≡ [(1 - 2λ), (2 - λ), (3 + λ)] Direction ratios of Normal of plane (2) = (a_2, b_2, c_2) ≡ (-5, -3, 6)

Since the two lines are perpendicular, a_1a_2
+ b_1b_2 + c_1c_2 = 0

(1 - 2λ) × (-5) + (2 - λ) × (-3) + (3 + λ) × 6 = 0

-5 + 10λ - 6 + 3λ + 18 + 6λ = 0

19λ + 7 = 0 λ
= -7/19

By substituting the value of λ in equation (1), we get

$$\bar{r}.\left((1-2\lambda)\hat{i}+(2-\lambda)\hat{j}+(3+\lambda)\hat{k}\right)=4+5\lambda$$

$$\bar{r}.\left(\left(1-2\left(\frac{-7}{19}\right)\right)\hat{i}+\left(2-\left(\frac{-7}{19}\right)\right)\hat{j}+\left(3+\left(\frac{-7}{19}\right)\right)\hat{k}\right)=4+5\left(\frac{-7}{19}\right)$$

$$\bar{r}.\left(\frac{33}{19}\hat{i}+\frac{45}{19}\hat{j}+\frac{50}{19}\hat{k}\right)=\frac{41}{19}$$

$$\vec{r}.\left(33\hat{i}+45\hat{j}+50\hat{k}\right)=41$$

∴ The equation of the required plane is $\vec{r}.\left(33\hat{i}+45\hat{j}+50\hat{k}\right)=41$

18. Find the distance of the point (–1, –5, –10) from the point of intersection of the line

$\vec{r} = \left(2\hat{i} - \hat{j} + 2\hat{k}\right) + \lambda\left(3\hat{i} + 4\hat{j} + 2\hat{k}\right)$ **and the plane** $\vec{r}.\left(\hat{i} - \hat{j} + \hat{k}\right) = 5$**.**

Solution:

Given:

The equation of line is

$$\vec{r} = \left(2\hat{i} - \hat{j} + 2\hat{k}\right) + \lambda\left(3\hat{i} + 4\hat{j} + 2\hat{k}\right) \quad \ldots (1)$$

And the equation of the plane is given by

$$\vec{r}.\left(\hat{i} - \hat{j} + \hat{k}\right) = 5 \quad \ldots (2)$$

Now to find the intersection of line and plane, substituting the value of $\vec{r}$ from equation (1) of line into equation of plane (2), we get

$$\left[\left(2\hat{i} - \hat{j} + 2\hat{k}\right) + \lambda\left(3\hat{i} + 4\hat{j} + 2\hat{k}\right)\right].\left(\hat{i} - \hat{j} + \hat{k}\right) = 5$$

$$\left[(2 + 3\lambda)\hat{i} + (-1 + 4\lambda)\hat{j} + (2 + 2\lambda)\hat{k}\right].\left(\hat{i} - \hat{j} + \hat{k}\right) = 5$$

$(2 + 3\lambda) \times 1 + (-1 + 4\lambda) \times (-1) + (2 + 2\lambda) \times 1 = 5$

$2 + 3\lambda + 1 - 4\lambda + 2 + 2\lambda = 5$

$\lambda = 0$

So, the equation of line is

$$\vec{r} = \left(2\hat{i} - \hat{j} + 2\hat{k}\right)$$

Let the point of intersection be (x, y, z)

So,

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$x\hat{i} + y\hat{j} + z\hat{k} = 2\hat{i} - \hat{j} + 2\hat{k}$$

Where, x = 2, y =
-1, z = 2

So, the point of intersection is (2, -1, 2).

Now, the distance between points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by

$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$ Units

Distance between the points A (2, -1, 2) and B (-1, -5, -10) is given by

$$AB = \sqrt{(2-(-1))^2 + (-1-(-5))^2 + (2-(-10))^2}$$

$$= \sqrt{(3)^2 + (4)^2 + (12)^2}$$

$$= \sqrt{9+16+144}$$

$$= \sqrt{169}$$

$$= 13 \text{ units}$$

$\therefore$ The distance is 13 units.

19. Find the vector equation of the line passing through (1, 2, 3) and parallel to the planes $\vec{r}.\left(\hat{i}-\hat{j}+2\hat{k}\right)=5$ **and** $\vec{r}.\left(3\hat{i}+\hat{j}+\hat{k}\right)=6$

Solution:

The vector equation of a line passing through a point with position vector $\vec{a}$ and parallel to a vector $\vec{b}$ is

$\vec{r}=\vec{a}+\lambda\vec{b}$

It is given that the line passes through (1, 2, 3)

So,

$\vec{a}=1\hat{i}+2\hat{j}+3\hat{k}$

It is also given that the line is parallel to both planes.

So line is perpendicular to normal of both planes.

i.e $\vec{b}$ is perpendicular to normal of both planes.

We know that

$\vec{a}\times\vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$

So, $\vec{b}$ is cross product of normal of plane $\vec{r}.\left(\hat{i}-\hat{j}+2\hat{k}\right)=5$ and $\vec{r}.\left(3\hat{i}+\hat{j}+\hat{k}\right)=6$

$$\text{Required Normal}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -1 & 2\\ 3 & 1 & 1\end{vmatrix}$$

$$=\hat{i}\left[(-1)(1)-1(2)\right]-\hat{j}\left[1(1)-3(2)\right]+\hat{k}\left[1(1)-3(-1)\right]$$

$$=\hat{i}[-1-2]-\hat{j}[1-6]+\hat{k}[1+3]$$

$$=-3\hat{i}+5\hat{j}+4\hat{k}$$

So,

$\vec{b}=-3\hat{i}+5\hat{j}+4\hat{k}$

Now, substitute the value of $\vec{a}$ & $\vec{b}$ in the formula, we get

$\vec{r}=\vec{a}+\lambda\vec{b}$

$$=\left(1\hat{i}+2\hat{j}+3\hat{k}\right)+\lambda\left(-3\hat{i}+5\hat{j}+4\hat{k}\right)$$

$\therefore$ The equation of the line is

$$\vec{r}=\left(1\hat{i}+2\hat{j}+3\hat{k}\right)+\lambda\left(-3\hat{i}+5\hat{j}+4\hat{k}\right)$$

20. Find the vector equation of the line passing through the point (1, 2, – 4) and perpendicular to the two lines:

$$\frac{x-8}{3}=\frac{y+19}{-16}=\frac{z-10}{7}$$ **and** $$\frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5}.$$

Solution:

The vector equation of a line passing through a point with position vector $\vec{a}$ and parallel to a vector $\vec{b}$ is $\vec{r} = \vec{a} + \lambda \vec{b}$

It is given that, the line passes through (1, 2, -4)

So,

$\vec{a} = 1\hat{i} + 2\hat{j} - 4\hat{k}$

It is also given that, line is parallel to both planes.

So we can say that the line is perpendicular to normal of both planes.

i.e $\vec{b}$ is perpendicular to normal of both planes.

We know that

$\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ & $\vec{b}$

So, $\vec{b}$ is cross product of normal of planes

$\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7}$ and $\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}$

$$\text{Required Normal} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -16 & 7 \\ 3 & 8 & -5 \end{vmatrix}$$

$$= \hat{i}\left[(-16)(-5) - 8(7)\right] - \hat{j}\left[3(-5) - 3(7)\right] + \hat{k}\left[3(8) - 3(-16)\right]$$

$$= \hat{i}[80 - 56] - \hat{j}[-15 - 21] + \hat{k}[24 + 48]$$

$$= 24\hat{i} + 36\hat{j} + 72\hat{k}$$

So,

$$\vec{b} = 24\hat{i} + 36\hat{j} + 72\hat{k}$$

Now, by substituting the value of $\vec{a}$ & $\vec{b}$ in the formula, we get

$$\vec{r} = \vec{a} + \lambda\,\vec{b}$$

$$= \left(1\hat{i} + 2\hat{j} - 4\hat{k}\right) + \lambda\left(24\hat{i} + 36\hat{j} + 72\hat{k}\right)$$

$$= \left(1\hat{i} + 2\hat{j} - 4\hat{k}\right) + 12\,\lambda\left(2\hat{i} + 3\hat{j} + 6\hat{k}\right)$$

$$= \left(1\hat{i} + 2\hat{j} - 4\hat{k}\right) + \lambda\left(2\hat{i} + 3\hat{j} + 6\hat{k}\right)$$

∴ The equation of the line is

$$\vec{r} = \left(1\hat{i} + 2\hat{j} - 4\hat{k}\right) + \lambda\left(2\hat{i} + 3\hat{j} + 6\hat{k}\right)$$

21. Prove that if a plane has the intercepts a, b, c and is at a distance of p units from the origin, then

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{p^2}$$

Solution:

We know that the distance of the point (x_1, y_1, z_1) from the plane Ax + By + Cz = D is given as

$$\left|\frac{Ax_1 + By_1 + Cz_1 - D}{\sqrt{A^2 + B^2 + C^2}}\right|$$

The equation of a plane having intercepts a, b, c on the x-, y-, z- axis respectively is given as

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

Let us compare it with Ax + By + Cz = D, we get

A = 1/a, B = 1/b, C = 1/c, D = 1

It is given that, the plane is at a distance of 'p' units from the origin.

So, the origin point is O (0, 0, 0)

Where, $x_1 = 0$, $y_1 = 0$, $z_1 = 0$

Now,

$$\text{Distance} = \left|\frac{Ax_1 + By_1 + Cz_1 - D}{\sqrt{A^2 + B^2 + C^2}}\right|$$

By substituting all values in above equation, we get

$$p = \left|\frac{\frac{1}{a}\times 0 + \frac{1}{b}\times 0 + \frac{1}{c}\times 0 - 1}{\sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2 + \left(\frac{1}{c}\right)^2}}\right|$$

$$p = \left|\frac{0+0+0-1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}}\right|$$

$$p = \left|\frac{-1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}}\right|$$

$$p = \frac{1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}}$$

$$\frac{1}{p} = \sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}$$

Now let us square on both sides, we get

$$\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$$

Hence Proved.

22. Distance between the two planes: $2x + 3y + 4z = 4$ and $4x + 6y + 8z = 12$ is A. 2 units

B. 4 units
C. 8 units
D. 2/√29 units Solution:
We know that the distance between two parallel planes Ax + By + Cz = d_1 and Ax + By + Cz = d_2 is given as

$$\left|\frac{d_1 - d_2}{\sqrt{A^2 + B^2 + C^2}}\right|$$

It is given that:
First Plane:
2x + 3y + 4z = 4
Let us compare with Ax + By + Cz = d_1, we get
A = 2, B = 3, C = 4, d_1 = 4

Second Plane:
4x + 6y + 8z = 12 [Divide the equation by 2]
We get,
2x + 3y + 4z = 6
Now comparing with Ax + By + Cz = d_1, we get
A = 2, B = 3, C = 4, d_2 = 6

So,
Distance between two planes is given as

$$= \left|\frac{4-6}{\sqrt{2^2+3^2+4^2}}\right|$$

$$= \left|\frac{-2}{\sqrt{4+9+16}}\right|$$

= 2/√29
∴ Option (D) is the correct option.

23. The planes: 2x – y + 4z = 5 and 5x – 2.5y + 10z = 6 are
A. Perpendicular
B. Parallel

C. intersect y–axis D. passes through Solution:

It is given that:

First Plane:

2x – y + 4z = 5 [Multiply both sides by 2.5]

We get,

5x - 2.5y + 10z = 12.5 … (1)

Given second Plane:

5x – 2.5y + 10z = 6 … (2)

So,

$$\frac{a_1}{a_2} = \frac{2}{5}$$

$$\frac{b_1}{b_2} = \frac{2}{5}$$

$$\frac{c_1}{c_2} = \frac{4}{10} = \frac{2}{5}$$

Hence,

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

It is clear that the direction ratios of normal of both the plane (1) and (2) are same. ∴ Both the given planes are parallel.